Point-Set Topology: Lecture 11

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1 Bases and Subbases

The topology on a set X generated by a collection of subsets $B \subseteq \mathcal{P}(X)$ gives us the notion of *subbasis*.

Definition 1.1 (**Subbasis of a Topology**) A subbasis of a topology τ on a set X is a set $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\tau = \tau(\mathcal{B})$, where $\tau(\mathcal{B})$ is the topology generated by $\mathcal{B}$. ■

A comment on this definition. Some require that $\mathcal{B}$ also covers the set X. That is, for all $x \in X$ there is some $\mathcal{U} \in \mathcal{B}$ such that $x \in \mathcal{U}$. To me this is slightly superfluous. If you are given a topology τ and a collection $\mathcal{B}$ such that $\tau(\mathcal{B}) = \tau$, you can get a new collection $\tilde{\mathcal{B}}$ that covers X via $\tilde{\mathcal{B}} = \mathcal{B} \cup \{\, X \,\}$. That is, take your original collection and just throw the entire set in. Since a topology requires the whole space to be open, we see that $\tau(\mathcal{B}) = \tau(\tilde{\mathcal{B}})$. That is, the topology generated by $\mathcal{B}$ is the same as the topology generated by $\tilde{\mathcal{B}}$. The benefits of requiring or omitting this new constraint are scarce. On the one hand, you can now say that the empty set serves as a subbasis of the indiscrete topology since $\tau(\emptyset) = \{\, \emptyset,\, X \,\}$. On the other, you may want a subbasis to also serve as an open cover in a theorem, and it may be nice to not have to explicitly say that the subbasis is an open cover every time.

With a subbasis you take a collection of subsets of X and declare that you *want* these sets to be open. The topology from this subbasis $\mathcal{B}$ is the *smallest* topology that contains $\mathcal{B}$ as a subset. This is done to define many new topological spaces that we can't easily define explicitly using a formula or rule for the open sets.

Example 1.1 The standard topology $\tau_{\mathbb{R}}$ on $\mathbb{R}$, the metric topology from

$$d(x,\, y) = |x - y| \tag{1}$$

has as a subbasis the collection of all open intervals. Let $\mathcal{B}$ be defined by:

$$\mathcal{B} = \{\, (a,\, b) \subseteq \mathbb{R} \mid a, b \in \mathbb{R} \text{ and } a < b \,\} \tag{2}$$

The topology generated by this set is the standard topology on $\mathbb{R}$. This collection $\mathcal{B}$ is **not** a topology. It lacks the union criterion. If $a, b, c, d \in \mathbb{R}$ and $a < b <$

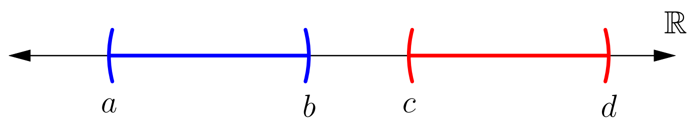

Figure 1: The Union of Open Intervals of the Real Line

$c < d$, the open intervals $(a,\, b)$ and $(c,\, d)$ do not overlap, meaning the union $(a,\, b) \cup (c,\, d)$ is two disjoint open intervals, which is not itself an open interval. This is shown in Fig. 1. ■

Example 1.2 (**The Countable Extension Topology**) Let τ_C by the countable complement topology on $\mathbb{R}$ and $\tau_{\mathbb{R}}$ the standard Euclidean topology. The countable extension topology is the topology $\tau_E = \tau(\tau_C \cup \tau_{\mathbb{R}})$. That is, the topology generated by the union of the countable complement and standard topologies. This space is not easy to describe explicitly in terms of what the open sets are precisely, but it is easy to say what a subbasis is. All open intervals and all sets whose complement is countable create a subbasis for this space. This space serves as a counterexample to the claim *Hausdorff implies metrizable*. The countable complement extension topology is Hausdorff since $\tau_{\mathbb{R}} \subseteq \tau_E$ and $\tau_{\mathbb{R}}$ is Hausdorff, but it is not metrizable. This space lacks a lot of the properties of metrizable spaces. It is not first countable, not regular, not normal, not perfectly normal, and not paracompact. We'll discuss all of these ideas soon enough. ■

A stronger notion than subbasis is that of a basis. Bases in topological spaces are very similar to bases in vector spaces. A basis is a collection of open sets that *spans* the topology. Every element of the topology can be written as the *sum* (union) of elements of the basis.

Definition 1.2 (**Basis for a Topology**) A basis for a topology τ on a set X is a set $\mathcal{B} \subseteq \tau$ such that $\mathcal{B}$ is an open cover of $(X,\, \tau)$ that generates τ and for all $\mathcal{U}, \mathcal{V} \in \mathcal{B}$ and for all $x \in \mathcal{U} \cap \mathcal{V}$ there is a $\mathcal{W} \in \mathcal{B}$ such that $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$. ■

Theorem 1.1. *If $(X,\, \tau)$ is a topological space and $\mathcal{B} \subseteq \tau$, then $\mathcal{B}$ is a basis if and only if for all $\mathcal{U} \in \tau$ there is an $\mathcal{O} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{O} = \mathcal{U}$.*

Proof. Suppose $\mathcal{B} \subseteq \tau$ is such that for all $\mathcal{U} \in \tau$ there is an $\mathcal{O} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{O} = \mathcal{U}$. Setting $\mathcal{U} = X$ shows that $\mathcal{B}$ is an open cover of $(X,\, \tau)$, the first property of a basis. Given $\mathcal{U}, \mathcal{V} \in \mathcal{B}$, and any $x \in \mathcal{U} \cap \mathcal{V}$, since $\mathcal{U} \cap \mathcal{V}$ is open there is some $\mathcal{O} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{O} = \mathcal{U} \cap \mathcal{V}$. But since $x \in \mathcal{U} \cap \mathcal{V}$, and since $\bigcup \mathcal{O} = \mathcal{U} \cap \mathcal{V}$, there must be some $\mathcal{W} \in \mathcal{O}$ such that $x \in \mathcal{W}$. But then $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$, the third property of a basis. Lastly we must show the topology generated by $\mathcal{B}$ is indeed τ. Since every element of τ can be written as the union of elements in $\mathcal{B}$, we see that the topology generated by $\mathcal{B}$ contains τ. But τ

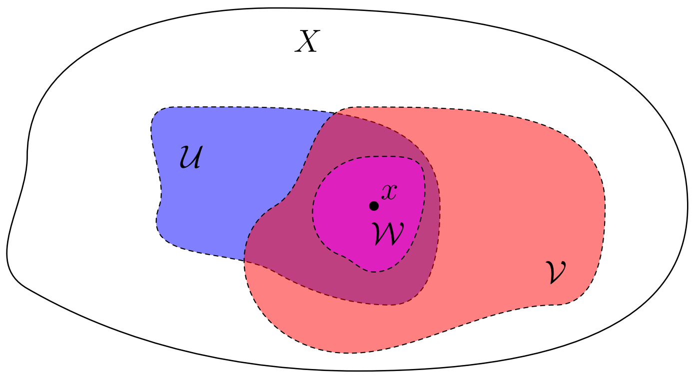

Figure 2: Condition for a Basis

itself is a topology that contains $\mathcal{B}$, and hence τ contains the topology generated by $\mathcal{B}$. We conclude that τ is identical to the topology generated by $\mathcal{B}$.

Now, let τ' be the set of all unions of subsets of $\mathcal{B}$. That is:

$$\tau' = \{ \bigcup \mathcal{O} \mid \mathcal{O} \subseteq B \} \tag{3}$$

If we can prove $\tau = \tau'$, we are done, since then every open set $\mathcal{U} \in \tau$ can be written as the union of elements of $\mathcal{B}$. We do this by proving that τ' is a topology, and that every topology containing $\mathcal{B}$ must have τ' as a subset, and hence the topology generated by $\mathcal{B}$ is precisely τ'. Since $\mathcal{B}$ generates τ, we then see that $\tau = \tau'$. To start, $\emptyset \in \tau'$ since we can choose $\mathcal{O} = \emptyset$. Next, since $\mathcal{B}$ is an open cover, choosing $\mathcal{O} = \mathcal{B}$ we see that $X \in \tau'$. Suppose $\mathcal{U}, \mathcal{V} \in \tau'$. Then there are $\mathcal{O}_{\mathcal{U}}, \mathcal{O}_{\mathcal{V}} \subseteq \mathcal{B}$ such that $\mathcal{U} = \bigcup \mathcal{O}_{\mathcal{U}}$ and $\mathcal{V} = \bigcup \mathcal{O}_{\mathcal{V}}$. Let $\mathcal{O} = \mathcal{O}_{\mathcal{U}} \cap \mathcal{O}_{\mathcal{V}}$. Then $\bigcup \mathcal{O} \subseteq \mathcal{U} \cap \mathcal{V}$. Let's reverse this. If $x \in \mathcal{U} \cap \mathcal{V}$, then $x \in \mathcal{U}$ and $x \in \mathcal{V}$, and hence there is some $\mathcal{W}_{\mathcal{U}} \in \mathcal{O}_{\mathcal{U}}$ such that $x \in \mathcal{W}_{\mathcal{U}}$ and some $\mathcal{W}_{\mathcal{V}} \in \mathcal{O}_{\mathcal{V}}$ such that $x \in \mathcal{W}_{\mathcal{V}}$. But then $\mathcal{W}_{\mathcal{U}}$ and $\mathcal{W}_{\mathcal{V}}$ are elements of $\mathcal{B}$, and x is common to their intersection, so there is some $\mathcal{W} \in \mathcal{B}$ such that $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{W}_{\mathcal{U}} \cap \mathcal{W}_{\mathcal{V}}$. But then $\mathcal{W} \in \mathcal{O}_{\mathcal{U}}$ and $\mathcal{W} \in \mathcal{O}_{\mathcal{V}}$, so $\mathcal{W} \in \mathcal{O}$, and therefore $x \in \bigcup \mathcal{O}$. Thus we have shown that $\mathcal{U} \cap \mathcal{V} \subseteq \bigcup \mathcal{O}$ and we may conclude that the two sets are indeed equal. τ' is therefore closed to finite intersections. Lastly, arbitrary unions. If $\mathcal{O} \subseteq \tau'$, for all $\mathcal{U} \in \mathcal{O}$ there is some $\mathcal{O}_{\mathcal{U}} \subseteq \mathcal{B}$ such that $\mathcal{U} = \bigcup \mathcal{O}_{\mathcal{U}}$. By collecting all elements of $\mathcal{B}$ belonging to one of these sets we form a new set $\mathcal{O}' \subseteq \mathcal{B}$ with the property that $\bigcup \mathcal{O}' = \bigcup \mathcal{O}$, and hence τ' is closed under arbitrary unions. τ' is hence a topology. Since topologies must be closed under arbitrary unions, if τ'' is a topology containing $\mathcal{B}$ we see that $\tau' \subseteq \tau''$, and hence τ' is the topology *generated* by $\mathcal{B}$. Since τ is also the topology generated by $\mathcal{B}$, we have $\tau = \tau'$. Hence every element of τ can be written as the union of elements of $\mathcal{B}$. $\square$

Theorem 1.2. *If (X, τ) is a topological space, and if $\mathcal{B} \subseteq \tau$ is a basis, then for all $\mathcal{U} \subseteq X$, $\mathcal{U} \in \tau$ if and only if there is an $\mathcal{O} \subseteq \mathcal{B}$ such that $\mathcal{U} = \bigcup \mathcal{O}$.*

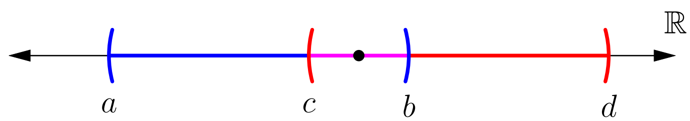

Figure 3: Open Intervals in $\mathbb{R}$ Form a Basis

Proof. The previous theorem shows that if $\mathcal{B}$ is a basis, then we can write $\mathcal{U} \in \tau$ via $\mathcal{U} = \bigcup \mathcal{O}$ for some $\mathcal{O} \subseteq \mathcal{B}$. In the other direction, if $\mathcal{U} = \bigcup \mathcal{O}$ for some $\mathcal{O} \subseteq \mathcal{B}$, then since $\mathcal{B}$ is a basis it is a subset of τ, and hence $\mathcal{U}$ is the union of a collection of open sets which is therefore open. □

Example 1.3 Open intervals in $\mathbb{R}$ form a basis, as well as a subbasis, for the standard topology. Indeed, every basis is also a subbasis for any topology τ on a set X. Given two open sets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}$ and a point $x \in \mathcal{U} \cap \mathcal{V}$, since $\mathcal{U} \cap \mathcal{V}$ is open, there is some $\varepsilon > 0$ such that for all $y \in \mathbb{R}$, $|x - y| < \varepsilon$ implies $y \in \mathcal{U} \cap \mathcal{V}$. That is, the open interval $(x - \varepsilon,\, x + \varepsilon)$ sits inside the set $\mathcal{U} \cap \mathcal{V}$, showing that the collection of open intervals forms a basis for the topology of $\mathbb{R}$. This is easier to picture if $\mathcal{U} = (a,\, b)$ and $\mathcal{V} = (c,\, d)$ with $a < b$, $c < b$, and $c < d$. The intersection of $(a,\, b)$ and $(c,\, d)$ is the open interval $(c,\, b)$ (See Fig. 3). Given any point $x \in (c,\, d)$, the interval $(c,\, b)$ is a basis element and fits inside of $(a,\, b) \cap (c,\, d)$. ■

Example 1.4 If $(X,\, \tau)$ is a metrizable space, and if d is a metric on X such that $\tau = \tau_d$, then the set of all open balls in $(X,\, d)$ centered at all points of all radii forms a basis. That is, we may define:

$$\mathcal{B} = \{\, B_r^{(X,\, d)}(x) \subseteq X \mid x \in X \text{ and } r > 0 \,\} \tag{4}$$

The set $\mathcal{B}$ is a basis for the topology $\tau = \tau_d$. ■

Example 1.5 (**The Radial Interval Topology**) Let $X = \mathbb{R}^2$ be the Cartesian plane. The radial interval topology is defined on X by giving it the following basis $\mathcal{B}$. If L is an open line segment that does not include the origin but is contained on a line that passes through the origin, then $L \in \mathcal{B}$. If $\mathcal{U} \subseteq \mathbb{R}^2$ is a collection of open line segments through the origin, each of which contains the origin, then $\mathcal{U} \in \mathcal{B}$. The set $\mathcal{B}$ is a basis for a topology, and this topology τ_R is the *radial interval topology* on the Cartesian plane. It definitely has the feeling of the Paris plane, but it is not. If we let τ_P be the topology of the Paris plane, the topology induced by the Paris metric d_P, then $\tau_P \subsetneq \tau_R$. This inclusion does not reverse. Take the open interval in the x axis between the points $(-1,\, 0)$ and $(1,\, 0)$ in the plane. This contains the origin and is an open interval, so it is included in the basis $\mathcal{B}$, and hence is included in the topology τ_R. However, open balls about the origin in the Paris plane are disks, just like in the Euclidean plane. So this open interval containing the origin is not open in the Paris plane. This example serves as a counterexample to the following claim. If $(X,\, \tau)$ is a metrizable topological space, and if $(X,\, \tilde{\tau})$ is a topological space such that $\tau \subseteq \tilde{\tau}$, then $(X, \tilde{\tau})$ is metrizable. This is **false**. Ideas like this occur quite often. If $\tau \subseteq \tilde{\tau}$ and (X, τ) is Hausdorff, then $(X, \tilde{\tau})$ is Hausdorff, this is true. It is natural to think a similar claim might hold for metrizable spaces, but it does not. The Paris plane is metrizable, it comes from the Paris metric. The radial plane is not metrizable, even though $\tau_P \subseteq \tau_R$. ■