# Point-Set Topology: Lecture 13

Ryan Maguire

Summer 2022

## 1 First Countable

So far we've seen that sequential spaces are the nice ones (and Hausdorff spaces. Sequential *and* Hausdorff is just the bee's knees). We also have no tools for determining if a space is sequential or not. One of the most mild conditions one can impose on a topological space is that it be *first-countable*. Most of the topological spaces studied are first-countable, with some crucial exceptions (the Zariski topology on $\mathbb{R}$ is not first-countable). We now proceed to describe this notion, give examples, prove some theorems, and show how first-countable spaces relate to sequential spaces.

**Definition 1.1** (**Neighborhood Basis**) A neighborhood basis in a topological space $(X,\,\tau)$ of a point $x\in X$ is a subset $\mathcal{B}\subseteq\tau$ such that for all $\mathcal{U}\in\mathcal{B}$ it is true that $x\in\mathcal{U}$, and for all $\mathcal{V}\in\tau$ such that $x\in\mathcal{V}$, there is a $\mathcal{U}\in\mathcal{B}$ with $\mathcal{U}\subseteq\mathcal{V}$. ■

First-countable spaces are defined in terms of neighborhood bases.

**Definition 1.2** (**First-Countable Topological Space**) A first-countable topological space is a topological space $(X,\,\tau)$ such that for all $x\in X$ there exists a countable subset $\mathcal{B}\subseteq\tau$ such that $\mathcal{B}$ is a neighborhood basis for $x$. ■

**Example 1.1** The real line with the standard topology is first-countable. Given $x\in\mathbb{R}$, define $\mathcal{U}_n=(x-\frac{1}{n+1},\,x+\frac{1}{n+1})$. The set $\mathcal{B}=\{\,\mathcal{U}_n\mid n\in\mathbb{N}\,\}$ forms a neighborhood basis for $x$. Given any open $\mathcal{V}\subseteq\mathbb{R}$ with $x\in\mathcal{V}$, there is an $\varepsilon>0$ such that for all $y\in\mathbb{R}$ with $|x-y|<\varepsilon$, it is true that $y\in\mathcal{V}$. Choosing $n$ so that $\frac{1}{n+1}<\varepsilon$ shows that $\mathcal{U}_n\subseteq\mathcal{V}$. ■

**Example 1.2** All Euclidean spaces are first-countable as well. We can apply a similar trick in $\mathbb{R}^n$ by surrounding $\mathbf{x}\in\mathbb{R}^n$ with countably many open balls of radius $\frac{1}{n+1}$. ■

The ideas behind these two examples only require the existence of open sets that get *smaller*, in a sense. Metric spaces have such a notion, *open balls*.

**Theorem 1.1.** *If $(X,\,\tau)$ is a metrizable topological space, then it is first-countable.*

*Proof.* Since $(X,\,\tau)$ is metrizable, there is a metric $d$ on $X$ such that $\tau=\tau_d$, where $\tau_d$ is the metric topology. Let $x\in X$ and define:

$$\mathcal{B}=\{\,B_{\frac{1}{n+1}}^{(X,\,d)}(x)\mid n\in\mathbb{N}\,\}\tag{1}$$

$\mathcal{B}$ is a countable neighborhood basis of $x$. First, since $\frac{1}{n+1}>0$ for all $n\in\mathbb{N}$, these open balls are non-empty and all contain the point $x$. Next, the set is countable since the elements are indexed by the natural numbers. Lastly, given any $\mathcal{U}\in\tau_d$ with $x\in\mathcal{U}$, by the definition of the metric topology there is an $\varepsilon>0$ such that:

$$B_{\varepsilon}^{(X,\,d)}(x)\subseteq\mathcal{U}\tag{2}$$

Choose $n\in\mathbb{N}$ such that $\frac{1}{n+1}<\varepsilon$. Then:

$$B_{\frac{1}{n+1}}^{(X,\,d)}(x)\subseteq B_{\varepsilon}^{(X,\,d)}(x)\subseteq\mathcal{U}\tag{3}$$

hence $\mathcal{B}$ is a countable neighborhood basis of $x$, so $(X,\,\tau)$ is first-countable. □

First-countable always implies sequential. It is often easier to prove a particular space is first-countable rather than prove it is sequential, so the following theorem has many uses in topology.

**Theorem 1.2.** *If $(X,\,\tau)$ is a first-countable topological space, then it is sequential.*

*Proof.* Suppose not. Then there is $\mathcal{U}\subseteq X$ that is sequentially open but not open. Since $\mathcal{U}$ is not open, there is an $x\in\mathcal{U}$ such that for all $\mathcal{V}\in\tau$ with $x\in\mathcal{V}$, it is not true that $\mathcal{V}\subseteq\mathcal{U}$ (otherwise we could write $\mathcal{U}$ as the union of all such $\mathcal{V}_x$, which is the union of open sets, and hence open, but $\mathcal{U}$ is not open). Since $(X,\,\tau)$ is first countable, there is a countable neighborhood basis $\mathcal{B}$ of $x$. Since $\mathcal{B}$ is countable, there is a surjection $\mathcal{V}:\mathbb{N}\rightarrow\mathcal{B}$. That is, we can list the elements of $\mathcal{B}$ as:

$$\mathcal{B}=\{\,\mathcal{V}_0,\,\dots,\,\mathcal{V}_n,\,\dots\,\}\tag{4}$$

Define $\mathcal{W}_n$ via:

$$\mathcal{W}_n=\bigcap_{k=0}^{n}\mathcal{V}_k\tag{5}$$

For all $n\in\mathbb{N}$, $\mathcal{W}_n$ is open since it is the finite intersection of open sets. But $x\in\mathcal{W}_n$, meaning there is an $a_n\in\mathcal{W}_n$ such that $a_n\notin\mathcal{U}$ (again, since $\mathcal{U}$ is not open and $x\in\mathcal{U}$ was chosen so that there are no open subsets of $X$ that contain $x$ and fit inside $\mathcal{U}$). We must show $a_n\rightarrow x$. Let $\mathcal{E}\in\tau$ be an open set with $x\in\mathcal{E}$. Since $\mathcal{B}$ is a neighborhood basis, there is an $N\in\mathbb{N}$ such that $\mathcal{V}_N\subseteq\mathcal{E}$. But then for all $n>N$, by the definition of $\mathcal{W}_n$, we have that $\mathcal{W}_n\subseteq\mathcal{V}_N$ and hence $\mathcal{W}_n\subseteq\mathcal{E}$. But then for all $n>N$ we have $a_n\in\mathcal{E}$. Hence, $a_n\rightarrow x$. But $\mathcal{U}$ is sequentially open and $x\in\mathcal{U}$, so if $a_n\rightarrow x$, then there is an $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ with $n>N$ we have $a_n\in\mathcal{U}$. But by definition of the sequence, for all $n\in\mathbb{N}$, $a_n\notin\mathcal{U}$, a contradiction. Hence, $\mathcal{U}$ is open and $(X,\,\tau)$ is sequential. □

## 2 Second Countable

Second-countable is a much stronger notion than first-countable. All of the spaces the human brain can hope to visualize are second-countable. Spaces we can envision are subsets of $\mathbb{R}^n$ for some $n \in \mathbb{N}$ (in fact, probably just 0, 1, 2, 3, and *maybe* 4 if you're really good). In other words, metric subspaces of Euclidean spaces. Any metric subspace of $\mathbb{R}^n$ is second-countable, so if you are to try and picture a topological space (without lying to yourself), the space better be second-countable. With that, I give a definition.

**Definition 2.1** (**Second-Countable Topological Space**) A second-countable topological space is a topological space $(X,\,\tau)$ such that there exists a countable basis $\mathcal{B}$ for the topology $\tau$. ■

Second-countable spaces are *not too big.*

**Example 2.1** The real line, with the standard topology, is second-countable. Let $\mathcal{B}$ be the set of all intervals $(a,\,b)$ with $a, b \in \mathbb{Q}$ and $a < b$. This set has the cardinality of $\mathbb{Q} \times \mathbb{Q}$, which is countable. It is also a basis, essentially because the rationals are dense in $\mathbb{R}$. So $\mathcal{B}$ is a countable basis for the real line, meaning $\mathbb{R}$ is second-countable. ■

**Example 2.2** All Euclidean spaces are second-countable. We can apply a similar trick, take all points $\mathbf{x} \in \mathbb{R}^n$ where every coordinate in $\mathbf{x}$ is rational. That is, if $\mathbf{x} = (x_0,\,\dots,\,x_{n-1})$, require that $x_k \in \mathbb{Q}$ for all $k \in \mathbb{Z}_n$. About each point collect all open balls with rational radii. This set has cardinality $\mathbb{Q}^{n+1}$, which again is countable (essentially by induction). This also forms a basis, showing that $\mathbb{R}^n$ is second-countable. ■

**Example 2.3** Not every metrizable space is second-countable. Equip $\mathbb{R}$ with the discrete topology $\mathcal{P}(\mathbb{R})$. Since all of the singletons $\{\,x\,\}$ are open, any basis $\mathcal{B}$ must include a copy of $\{\,x\,\}$ for each $x \in \mathbb{R}$, meaning $\mathcal{B}$ cannot be countable since $\mathbb{R}$ is uncountable. The space is metrizable, however, since it comes from the discrete metric. ■

**Theorem 2.1.** *If $(X,\,\tau)$ is second-countable, then it is first-countable.*

*Proof.* Since $(X,\,\tau)$ is second-countable, there is a countable basis $\mathcal{B}$. Given $x \in X$, define $\mathcal{B}_x$ via:

$$\mathcal{B}_x = \{\,\mathcal{U} \in \mathcal{B} \mid x \in \mathcal{U}\,\} \tag{6}$$

Since $\mathcal{B}_x$ is a subset of a countable set, it is countable. It is also a neighborhood basis of $x$. By definition every element of $\mathcal{B}_x$ contains $x$. Suppose $\mathcal{V} \in \tau$ is an open set such that $x \in \mathcal{V}$. Since $\mathcal{B}$ is a basis, there is a $\mathcal{U} \in \mathcal{B}$ such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{V}$. But since $x \in \mathcal{U}$ it is true that $\mathcal{U} \in \mathcal{B}_x$. But then $\mathcal{U}$ is an element of $\mathcal{B}_x$ such that $\mathcal{U} \subseteq \mathcal{V}$. Hence, $(X,\,\tau)$ is first-countable. □

The converse does not hold. We've shown that all metrizable spaces are first-countable, but the discrete topology on $\mathbb{R}$ is a metrizable space that is not second-countable. That is, $\big(\mathbb{R},\,\mathcal{P}(\mathbb{R})\big)$ is a first-countable space that is not second-countable.

## 3 Separable

Separable spaces are important in analysis and geometry. The real line has the property that there is a countable subset $\mathbb{Q}$ that can approximate every point on the line. We take this idea to motivate the term separable.

**Definition 3.1** (**Separable Topological Space**) A separable topological space is a topological space $(X,\,\tau)$ such that there exists a countable dense subset $A\subseteq X$. That is, $A$ is countable and $\mathrm{Cl}_\tau(A)=X$. ■

**Example 3.1** The real line is separable, taking $A=\mathbb{Q}$ gives us a countable dense subset. $\mathbb{R}^n$ is also separable, setting $A=\mathbb{Q}^n$ shows there is a countable dense subset of $\mathbb{R}^n$. ■

The two previous examples of separable spaces are also second-countable. This is not a coincidence, every second-countable space is separable.

**Theorem 3.1.** *If $(X,\,\tau)$ is second-countable, then it is separable.*

*Proof.* Since $(X,\,\tau)$ is second-countable, there exists a countable basis $\mathcal{B}'$ for $\tau$. Let $\mathcal{B}\subseteq\mathcal{B}'$ be the set of all non-empty elements of $\mathcal{B}'$. Since $\mathcal{B}'$ is a countable basis, so is $\mathcal{B}$ (we're only removing the empty set). $\mathcal{B}$ now has the property that for all $\mathcal{V}\in\mathcal{B}$, $\mathcal{V}$ is non-empty. Since $\mathcal{B}$ is countable, there is a surjection $\mathcal{U}:\mathbb{N}\rightarrow\mathcal{B}$. That is, we may list the elements as:

$$\mathcal{B}=\{\,\mathcal{U}_0,\,\dots,\,\mathcal{U}_n,\,\dots\,\}\tag{7}$$

Since $\mathcal{U}_n$ is non-empty for all $n\in\mathbb{N}$, by the axiom of countable choice we can find a sequence $a:\mathbb{N}\rightarrow X$ such that $a_n\in\mathcal{U}_n$ for all $n\in\mathbb{N}$. Define $A$ by:

$$A=\{\,a_n\in X\mid n\in\mathbb{N}\,\}\tag{8}$$

$A$ is countable since it is index by the natural numbers. It is also dense. For suppose not. A set $B$ is dense if and only if for every non-empty open subset $\mathcal{V}\in\tau$ the intersection $B\cap\mathcal{V}$ is non-empty. Since we are supposing $A$ is not dense, there must be a non-empty open subset $\mathcal{V}$ such that $A\cap\mathcal{V}$ is empty. But since $\mathcal{V}$ is open and $\mathcal{B}$ is a basis, there is a $\mathcal{U}_n\in\mathcal{B}$ such that $\mathcal{U}_n\subseteq\mathcal{V}$. But $a_n\in\mathcal{U}_n$, so $a_n\in\mathcal{V}$. But $a_n\in A$, which is a contradiction since $A\cap\mathcal{V}=\emptyset$. Hence, $A$ is dense. □

**Example 3.2** (**The Particular Point Topology**) This theorem does not reverse. There are topological spaces that are separable but not second-countable. The easiest example to describe is the *particular point topology* on $\mathbb{R}$. Define $\mathcal{U}\subseteq\mathbb{R}$ to be open in $\tau$ if and only if $\mathcal{U}=\emptyset$ or $0\in\mathcal{U}$. This space is separable since $A=\{\,0\,\}$ is a dense subset and it is certainly countable since it is finite. $A$ is dense since the only closed set containing $\{\,0\,\}$ is $\mathbb{R}$. Given any closed set $\mathcal{C}$, $\mathbb{R}\setminus\mathcal{C}$ is open, meaning either $\mathbb{R}\setminus\mathcal{C}$ is empty, or $0\in\mathbb{R}\setminus\mathcal{C}$. So if $\mathcal{C}$ is a closed set with $0\in\mathcal{C}$, it must be true that $\mathbb{R}\setminus\mathcal{C}=\emptyset$. Hence, the only closed set contain 0 is $\mathbb{R}$. So, $\mathrm{Cl}_\tau\big(\{\,0\,\}\big)=\mathbb{R}$. This space is not second-countable. Every set of the form $\{ 0, x \}$ for all $x \in \mathbb{R}$ is open, so a basis must contain a copy of each of these. The cardinality of such a basis must then be at least as big as $\mathbb{R}$, which is uncountable, meaning $(\mathbb{R}, \tau)$ is not second-countable. ■

**Theorem 3.2.** *If $(X, \tau)$ is metrizable and separable, then it is second-countable.*

*Proof.* Since $(X, \tau)$ is metrizable, there is a metric $d$ on $X$ such that $\tau = \tau_d$, where $\tau_d$ is the metric topology from $d$. Since $(X, \tau)$ is separable, there is a countable dense subset $A \subseteq X$. Define $\mathcal{B}$ via:

$$\mathcal{B} = \{ B_r^{(X,\, d)}(x) \mid x \in A \text{ and } q \in \mathbb{Q}^+ \} \tag{9}$$

That is, $\mathcal{B}$ is the set of all balls of rational radii centered at all points in $A$. Since $A$ and $\mathbb{Q}$ are countable, the set $\mathcal{B}$ is countable as well. $\mathcal{B}$ is a basis. It covers $X$, for given $y \in X$, pick any $x \in A$ and $r \in \mathbb{Q}^+$ such that $r > d(x, y)$. Then $y \in B_r^{(X,\, d)}(x)$, which is an element of $\mathcal{B}$, so $\mathcal{B}$ is an open cover of $(X, \tau)$. Suppose $\mathcal{U}, \mathcal{V} \in \tau$ and $x \in \mathcal{U} \cap \mathcal{V}$. Since $\mathcal{U} \cap \mathcal{V}$ is open and $x \in \mathcal{U} \cap \mathcal{V}$ there is an $r > 0$ such that $B_r^{(X,\, d)}(x) \subseteq \mathcal{U} \cap \mathcal{V}$. But since $r > 0$, $r/4 > 0$ as well, so $B_{r/4}^{(X,\, d)}(x)$ is open. But $A$ is dense, so there is a $y \in A$ such that $y \in B_{r/4}^{(X,\, d)}(x)$. Choose $\varepsilon \in \mathbb{Q}^+$ such that $r/4 < \varepsilon < r/2$. Then $B_\varepsilon^{(X,\, d)}(y) \subseteq B_r^{(X,\, d)}(x)$, so $B_\varepsilon^{(X,\, d)}(y) \subseteq \mathcal{U} \cap \mathcal{V}$ but also $x \in B_\varepsilon^{(X,\, d)}(y)$. That is, we've found an element of $\mathcal{B}$ that contains $x$ and fits inside of $\mathcal{U} \cap \mathcal{V}$. Hence, $\mathcal{B}$ is a countable basis. Therefore, $(X, \tau)$ is second countable. □

The trick *seems* to hint that a separable first-countable space should be second-countable, but this is **false**. The metric topology was very helpful in this proof in some subtle ways.

**Example 3.3** The particular point topology on $\mathbb{R}$ is separable (as we've already seen) and first-countable, but not second-countable (again, we saw this in a previous example). To show that it is first-countable, pick any $x \in \mathbb{R}$. The set $\mathcal{B} = \big\{ \{ 0, x \} \big\}$ is a neighborhood basis of $x$. Given any open $\mathcal{U} \subseteq \mathbb{R}$ that contains $x$, it must also contain 0 by the definition of the particular point topology. Hence $\{ 0, x \} \subseteq \mathcal{U}$. But also $\{ 0, x \}$ is open. This shows $\mathcal{B}$ is a neighborhood basis of $x$, and it is finite, hence countable. ■