Point-Set Topology: Lecture 14

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July 19, 2023

1 Homeomorphisms and Open Mappings

Thus far we've discussed conditions for when continuity can be described by sequences. It is worthwhile studying the general notion of continuity as well. As a reminder, given two topological spaces $(X,\,\tau_X)$ and $(Y,\,\tau_Y)$, a continuous function from X to Y is a function $f:X\to Y$ such that for all $\mathcal{V}\in\tau_Y$ it is true that $f^{-1}[\mathcal{V}]\in\tau_X$. That is, the pre-image of an open set is open. It was proved this is equivalent to the pre-image of a closed set being closed using some of the set-theoretic laws of pre-image and set difference. *Homeomorphism* is a stronger notion. It tells us when two topological spaces are the same.

Definition 1.1 (**Homeomorphism**) A homeomorphism from a topological space $(X,\,\tau_X)$ to a topological space $(Y,\,\tau_Y)$ is a bijective continuous function $f:X\to Y$ such that f^{-1} is continuous. ■

Example 1.1 Let $(X,\,\tau)$ be any topological space, and $f:X\to X$ be the identity function $f(x)=x$. Then f is a homeomorphism. It is certainly a bijection, but it is also continuous. Given $\mathcal{U}\in\tau$ we have $f^{-1}[\mathcal{U}]=\mathcal{U}$, which is an element of τ. Given $\mathcal{V}\in\tau$ we have $\big(f^{-1}\big)^{-1}[\mathcal{V}]=f[\mathcal{V}]=\mathcal{V}$, which is in τ (Note: $\big(f^{-1}\big)^{-1}[\mathcal{V}]=f[\mathcal{V}]=\mathcal{V}$ is true since f is a bijection). This shows f is a homeomorphism. ■

Example 1.2 Take $X=Y=\mathbb{R}$ and give both of these the standard Euclidean topology $\tau_{\mathbb{R}}$. The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^3$ is a homeomorphism. It is bijective, continuous, and the inverse is given by $f^{-1}(x)=\sqrt[3]{x}=x^{1/3}$, which is also continuous. ■

Example 1.3 Any continuous bijective function $f:\mathbb{R}\to\mathbb{R}$ is a homeomorphism with respect to the standard topology. This is **not** true for general topological spaces. **It is not true that a continuous bijection must have a continuous inverse**. The real line is special in this regard. This property comes from the fact that the real line has a complete total ordered (via the $<$ symbol). If $f:\mathbb{R}\to\mathbb{R}$ is a continuous bijection, it must be strictly increasing or strictly decreasing. If not, if there are $a<b<c$ with $f(a)<f(b)$ and $f(c)<f(b)$, or $f(b)<f(a)$ and $f(b)<f(c)$, then by the intermediate value theorem there must be values $x_0\in(a,\,b)$ and $x_1\in(b,\,c)$ such that $f(x_0)=f(x_1)$,

violating the fact that f is a bijection. Using this you can then show that f^{-1} is also continuous. ■

Example 1.4 Let $X = [0, 1)$ and $Y = \mathbb{S}^1 \subseteq \mathbb{R}^2$, the unit circle. Both of these are metric subspaces of the Euclidean spaces $\mathbb{R}$ and $\mathbb{R}^2$, respectively, meaning they are metric spaces in their own right, and hence topological spaces with the induced topology from the subspace metric. The function $f : [0, 1) \to \mathbb{S}^1$ defined by $f(t) = \big(\cos(2\pi t), \sin(2\pi t)\big)$ is a continuous bijection, but it is *not* a homeomorphism. To go from the circle to the interval requires *tearing* the circle at a point, and this operation is not continuous. ■

We can be more precice in proving that $[0, 1)$ and $\mathbb{S}^1$ do not have a homeomorphism between them. The idea of *compactness* from metric spaces is a notion that is preserved by homeomorphisms.

Theorem 1.1. *If (X, d_X) and (Y, d_Y) are metric spaces, if $f : X \to Y$ is a homeomorphism, and if (X, d_X) is compact, then (Y, d_Y) is compact.*

Proof. For let $b : \mathbb{N} \to Y$ be a sequence. Let $a : \mathbb{N} \to X$ be defined by $a_n = f^{-1}(b_n)$ (this is well-defined since f is a bijection and hence has an inverse). Since (X, d_X) is compact, there is a convergent subsequence a_k. Let $x \in X$ be the limit, $a_{k_n} \to x$. Then, since f is continuous, we have $f(a_{k_n}) \to f(x)$. But $f(a_{k_n}) = b_{k_n}$, and hence b_k is a convergent subsequence in Y, so (Y, d_Y) is compact. □

The circle $\mathbb{S}^1$ is compact by the Heine-Borel theorem since it is a closed and bounded subset of $\mathbb{R}^2$. The half-open interval $[0, 1)$ is not compact, again by Heine-Borel, since it is not closed. Since homeomorphisms preserve compactness, there can be no homeomorphism between $[0, 1)$ and $\mathbb{S}^1$.

Homeomorphisms give a notion of *equivalence* between topological spaces. There is no *set of all topological spaces*, just like there is no set of all sets, so it is meaningless to say there is an *equivalence relation on topological spaces*. Still, the following few theorems highlight what is meant by saying homeomorphisms tell us which spaces are equivalent.

Theorem 1.2. *If (X, τ) is a topological space, then there is a homeomorphism $f : X \to X$.*

Proof. Define $f : X \to Y$ by $f(x) = x$. Then f is bijective and continuous since $f^{-1}[\mathcal{V}] = \mathcal{V}$ for all $\mathcal{V} \in \tau$. The inverse is also continuous since $\big(f^{-1}\big)^{-1}[\mathcal{U}] = f[\mathcal{U}] = \mathcal{U}$ for all $\mathcal{U} \in \tau$, showing us that f^{-1} is continuous. So f is a homeomorphism. □

Theorem 1.3. *If (X, τ_X) and (Y, τ_Y) are topological spaces, and if $f : X \to Y$ is a homeomorphism, then there is a homeomorphism $g : Y \to X$.*

Proof. Define $g : Y \to X$ via $g(y) = f^{-1}(y)$. Since f is a homeomorphism it is bijective, meaning g is well-defined. But since f is bijective, f^{-1} is bijective, so g

is bijective. Since f^{-1} is continuous, g is continuous. Lastly, since $(f^{-1})^{-1} = f$, and f is continuous, it is true that g^{-1} is continuous. So $g : Y \to X$ is a homeomorphism. $\square$

To prove *transitivity*, we first need the following theorem.

Theorem 1.4. *If (X, τ_X), (Y, τ_Y), and (Z, τ_Z) are topological spaces, if $f : X \to Y$ is a continuous function, and if $g : Y \to Z$ is a continuous function, then $g \circ f : X \to Z$ is continuous.*

Proof. Let $\mathcal{W} \in \tau_Z$. Since g is continuous, $g^{-1}[\mathcal{W}] \in \tau_Y$. But since f is continuous, $f^{-1}\big[g^{-1}[\mathcal{W}]\big] \in \tau_X$. But $(g \circ f)^{-1}[\mathcal{W}] = f^{-1}\big[g^{-1}[\mathcal{W}]\big]$, so $g \circ f$ is continuous. $\square$

Theorem 1.5. *If (X, τ_X), (Y, τ_Y), and (Z, τ_Z) are topological spaces, if $f : X \to Y$ is a homeomorphism, and if $g : Y \to Z$ is a homeomorphism, then $g \circ f : X \to Z$ is a homeomorphism.*

Proof. The composition of bijections is a bijection, so $g \circ f$ is bijective. The composition of continuous functions is continuous, so $g \circ f$ is continuous. Lastly, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, which is the composition of continuous functions since f^{-1} and g^{-1} are continuous, so $(g \circ f)^{-1}$ is continuous. That is, $g \circ f$ is a homeomorphism. $\square$

Homeomorphisms are just rebalellings of topological spaces. Given (X, τ_X) and (Y, τ_Y), and a homeomorphism $f : X \to Y$, given $x \in X$ we *relabel* this as $f(x) = y \in Y$. Given $\mathcal{U} \in \tau_X$ we *relabel* this as $f[\mathcal{U}] = \mathcal{V} \in \tau_Y$. The real line $\mathbb{R}$ and the imaginary line $i\mathbb{R}$, which is the set of all complex numbers of the form iy with $y \in \mathbb{R}$, are topologically the same. They're just a line. We took a real number $r \in \mathbb{R}$ and relabelled it as $ir \in i\mathbb{R}$, but this doesn't really change anything. When we talk about the *initial* and *final* topologies in a few pages, this statement will be made clear.

Homeomorphisms preserve *topological properties*.

Theorem 1.6. *If (X, τ_X) is a Hausdorff topological space, if (Y, τ_Y) is a topological space, and if $f : X \to Y$ is a homeomorphism, then (Y, τ_Y) is Hausdorff.*

Proof. Let $y_0, y_1 \in Y$ with $y_0 \neq y_1$. Since f is a homeomorphism, it is bijective, so there exists unique $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. Since $y_0 \neq y_1$, we have $x_0 \neq x_1$. But (X, τ_X) is Hausdorff, so there exists $\mathcal{U}, \mathcal{V} \in \tau_X$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Let $\tilde{\mathcal{U}} = f[\mathcal{U}]$ and $\tilde{\mathcal{V}} = f[\mathcal{V}]$. Then, since f is a homeomorphism, it is bijective, and hence $\tilde{\mathcal{U}} = (f^{-1})^{-1}[\mathcal{U}]$ and $\tilde{\mathcal{V}} = (f^{-1})^{-1}[\mathcal{V}]$, the pre-image of open sets by a continuous function since f^{-1} is continuous, and hence $\tilde{\mathcal{U}}, \tilde{\mathcal{V}} \in \tau_Y$. But $y_0 \in \tilde{\mathcal{U}}$, $y_1 \in \tilde{\mathcal{V}}$, and $\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}} = \emptyset$ since:

$$\begin{aligned}
\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}} &= f[\mathcal{U}] \cap f[\mathcal{V}] && \text{(Substitution)} \\
&= f[\mathcal{U} \cap \mathcal{V}] && \text{(Since } f \text{ is bijective)} \\
&= f[\emptyset] && \text{(Since } \mathcal{U} \text{ and } \mathcal{V} \text{ are disjoint)} \\
&= \emptyset && \text{(The image of the empty set is empty)}
\end{aligned}$$

Hence (Y, τ_Y) is Hausdorff. □

If the target space (Y, τ_Y) is Hausdorff, and if $f : X \to Y$ is continuous and injective, you can then prove that (X, τ_X) is Hausdorff as well. You do not need f to be a homeomorphism in this direction.

Theorem 1.7. *If (X, τ_X) is a topological space, if (Y, τ_Y) is a Hausdorff topological space, and if $f : X \to Y$ is a continuous injective function, then (X, τ_X) is Hausdorff.*

Proof. Let $x_0, x_1 \in X$, $x_0 \neq x_1$. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$. Then since f is injective, $y_0 \neq y_1$. But (Y, τ_Y) is Hausdorff, so there exists $\mathcal{U}, \mathcal{V} \in \tau_Y$ such that $y_0 \in \mathcal{U}$, $y_1 \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. But f is continuous, so $f^{-1}[\mathcal{U}] \in \tau_X$ and $f^{-1}[\mathcal{V}] \in \tau_Y$. But then $x \in f^{-1}[\mathcal{U}]$, $y \in f^{-1}[\mathcal{V}]$, and:

$$f^{-1}[\mathcal{U}] \cap f^{-1}[\mathcal{V}] = f^{-1}[\mathcal{U} \cap \mathcal{V}] = f^{-1}[\emptyset] = \emptyset \tag{1}$$

so (X, τ_X) is Hausdorff. □

Many of the theorems about homeomorphisms use the fact that since f is bijective, $(f^{-1})^{-1}[\mathcal{U}] = \mathcal{U}$. So in particular, if $\mathcal{U} \subseteq X$ is open, then $f[\mathcal{U}] \subseteq Y$ is open. Functions with this property are called *open mappings*.

Definition 1.2 (**Open Mapping**) An open mapping from a topological space (X, τ_Y) to a topological space (Y, τ_Y) is a function $f : X \to Y$ such that for all $\mathcal{U} \in \tau_X$ it is true that $f[\mathcal{U}] \in \tau_Y$. ■

Open mappings do not need to be continuous and continuous functions do not need to be open mappings.

Example 1.5 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, with the standard Euclidean topology on $\mathbb{R}$. This is continuous, since it is a polynomial, but not an open mapping. The forward image of $(-1,\, 1)$ is $f\big[(-1,\, 1)\big] = [0,\, 1)$, which is not open. ■

Example 1.6 Let $X = Y = \mathbb{R}$, $\tau_X = \{\,\emptyset,\, \mathbb{R}\,\}$, and $\tau_Y = \mathcal{P}(\mathbb{R})$, and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. Then f is *not* continuous. The pre-image of $\{\,0\,\}$ is $\{\,0\,\}$. $\{\,0\,\}$ is an element of $\mathcal{P}(\mathbb{R})$, but not $\{\,\emptyset,\, \mathbb{R}\,\}$, so f is not continuous. f is an open mapping. There are only two elements of τ_X to check. We have $f[\emptyset] = \emptyset$ and $f[\mathbb{R}] = \mathbb{R}$, both of which are elements of τ_Y, so f is an open mapping. ■

Theorem 1.8. *If $(X,\, \tau_X)$, $(Y,\, \tau_Y)$, and $(Z,\, \tau_Z)$ are topological spaces, if $f : X \to Y$ and $g : Y \to Z$ are open mappings, then $g \circ f$ is an open mapping.*

Proof. Given $\mathcal{U} \in \tau_X$, we have:

$$\big(g \circ f\big)[\mathcal{U}] = g\big[f[\mathcal{U}]\big] \tag{2}$$

but since f is an open mapping, $f[\mathcal{U}] \in \tau_Y$. But since g is an open mapping, $g\big[f[\mathcal{U}]\big] \in \tau_Z$. Therefore $g \circ f$ is an open mapping. □

Theorem 1.9. *If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are topological spaces, and if $f : X \to Y$ is a function, then f is a homeomorphism if and only if f is continuous, bijective, and an open mapping.*

Proof. If f is a homeomorphism it is continuous and bijective. It is also an open mapping since if $\mathcal{U} \in \tau_X$, then:

$$f[\mathcal{U}] = (f^{-1})^{-1}[\mathcal{U}] \tag{3}$$

but f^{-1} is continuous since f is a homeomorphism, so $f[\mathcal{U}]$ is the pre-image of an open set under a continuous function and is therefore open. That is, f is an open mapping. Now suppose f is a continuous bijective open mapping. Let $\mathcal{U} \in \tau_X$. Then:

$$(f^{-1})^{-1}[\mathcal{U}] = f[\mathcal{U}] \tag{4}$$

But $\mathcal{U}$ is an open mapping, so $f[\mathcal{U}] \in \tau_Y$. Therefore f^{-1} is continuous and f is a homeomorphism. □

Homeomorphisms preserve the notion of *sequential* as well. First, the following theorem about sequentially open sets and continuous functions.

Theorem 1.10. *If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are topological spaces, if $f : X \to Y$ is continuous, and if $\mathcal{V} \subseteq Y$ is sequentially open, then $f^{-1}[\mathcal{V}]$ is sequentially open.*

Proof. Suppose not. Then there is an $x \in f^{-1}[\mathcal{V}]$ and a sequence $a : \mathbb{N} \to X$ such that $a_n \to x$ and for all $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $n > N$ and $a_n \notin f^{-1}[\mathcal{V}]$. But f is continuous, so it is sequentially continuous, and therefore $f(a_n) \to f(x)$. But since $x \in f^{-1}[\mathcal{V}]$ we have $f(x) \in \mathcal{V}$. But $\mathcal{V}$ is sequentially open, so if $f(a_n) \to f(x)$, then there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$ we have $f(a_n) \in \mathcal{V}$. But then for all $n > N$, $a_n \in f^{-1}[\mathcal{V}]$, a contradiction. So $f^{-1}[\mathcal{V}]$ is sequentially open. □

Theorem 1.11. *If $(X,\, \tau_X)$ is a sequential topological space, if $(Y,\, \tau_Y)$ is a topological space, and if $f : X \to Y$ is a homeomorphism, then $(Y,\, \tau_Y)$ is sequential.*

Proof. Let $\mathcal{V} \subseteq Y$ be sequentially open. Since f is a homeomorphism, f is continuous, so $f^{-1}[\mathcal{V}]$ is sequentially open. But $(X,\, \tau_X)$ is sequential, so if $f^{-1}[\mathcal{V}]$ is sequentially open, then it is open. But since f is a homeomorphism, it is an open mapping, meaning $f\big[f^{-1}[\mathcal{V}]\big] \in \tau_Y$. But f is bijective, so $f\big[f^{-1}[\mathcal{V}]\big] = \mathcal{V}$. That is, $\mathcal{V}$ is open, and $(Y,\, \tau_Y)$ is sequential. □

Theorem 1.12. *If $(X,\, \tau_X)$ is a second-countable topological space, if $(Y,\, \tau_Y)$ is a topological space, and if $f : X \to Y$ is a homeomorphism, then $(Y,\, \tau_Y)$ is second-countable.*

Proof. Since $(X,\, \tau_X)$ is second-countable, there is a countable basis $\mathcal{B}$. Define $\tilde{\mathcal{B}}$ by:

$$\tilde{\mathcal{B}} = \{\, f[\mathcal{U}] \mid \mathcal{U} \in \mathcal{B} \,\} \tag{5}$$

But f is a homeomorphism, so it is an open mapping, meaning for all $\mathcal{U} \in \tau_X$, $f[\mathcal{U}]$ is open, so $\tilde{\mathcal{B}} \subseteq \tau_Y$. Moreover since $\mathcal{B}$ is countable, so is $\tilde{\mathcal{B}}$. Let's show $\tilde{\mathcal{B}}$ is a basis. Given an open set $\mathcal{V} \in \tau_Y$, let $\mathcal{U} = f^{-1}[\mathcal{V}]$. Then $\mathcal{U} \in \tau_X$ since f is continuous, and since $\mathcal{B}$ is a basis there is some $\mathcal{O} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{O} = \mathcal{U}$. Define $\tilde{\mathcal{O}}$ via:

$$\tilde{\mathcal{O}} = \{\, f[\mathcal{W}] \mid \mathcal{W} \in \mathcal{O} \,\} \tag{6}$$

Then $\tilde{\mathcal{O}} \subseteq \tilde{\mathcal{B}}$, by the definition of $\tilde{\mathcal{B}}$, and since f is a bijection we may conclude that $\bigcup \tilde{\mathcal{O}} = f[\mathcal{U}] = \mathcal{V}$. Hence $\tilde{\mathcal{B}}$ is a countable basis for τ_Y. □

Similar to open mappings, one often studies *closed mappings*. Closed mappings arise quite frequently in functional analysis and geometry.

Definition 1.3 (**Closed Mappings**) A closed mapping from a topological space $(X,\, \tau_X)$ to a topological space $(Y,\, \tau_Y)$ is a function $f : X \to Y$ such that for every closed $\mathcal{C} \subseteq X$ it is true that $f[\mathcal{C}] \subseteq Y$ is closed. ■

Example 1.7 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is closed but not open. ■

Example 1.8 Give $X = Y = \mathbb{R}$, $\tau_X = \{\, \emptyset,\, \mathbb{R} \,\}$, and $\tau_Y = \mathcal{P}(\mathbb{R})$, the identity function $f(x) = x$ is closed (and open) but *not* continuous. ■

Open mappings, closed mappings, and continuous functions are three distinct notions.

Theorem 1.13. *If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are topological spaces, and if $f : X \to Y$ is a bijective open mapping, then it is a closed mapping.*

Proof. Let $\mathcal{C} \subseteq X$ be closed. Then $X \setminus \mathcal{C}$ is open. But then:

$$f[\mathcal{C}] = f[X \setminus (X \setminus \mathcal{C})] \tag{7}$$

But f is bijective, so:

$$f[X \setminus (X \setminus \mathcal{C})] = f[X] \setminus f[X \setminus \mathcal{C}] \tag{8}$$

But f is an open mapping, so $f[X \setminus \mathcal{C}]$ is open. But f is bijective, so $f[X] = Y$, and therefore $f[X] \setminus f[X \setminus \mathcal{C}]$ is the complement of an open set, which is therefore closed. Thus, f is a closed mapping. $\square$

Without bijectivity, this statement fails. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is a continuous closed mapping that is *not* an open mapping. The reason being that f is not bijective.

Homeomorphisms can also be described via closed mappings.

Theorem 1.14. *If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are topological spaces, and if $f : X \to Y$ is a function, then f is a homeomorphism if and only if f is continuous, bijective, and a closed mapping.*

Proof. f is a homeomorphism if and only if f is continuous, bijective, and open. If f is bijective, then f is open if and only if f is closed, so f is a homeomorphism if and only if f is continuous, bijective, and a closed mapping. $\square$

2 Subspaces

In the study of metric spaces, given such a space $(X,\, d)$ and a subset $A \subseteq X$, we could restrict the metric $d : X \times X \to \mathbb{R}$ to $d_A : A \times A \to \mathbb{R}$, making $(A,\, d_A)$ a metric space. This gave us a metric topology on A, and hence allowed us to think of A as a topological space, with the topology stemming from the metric topology on A. We then proved that a subset $\mathcal{U}$ of A is open with respect to this subspace topology if and only if there is an open subset $\mathcal{V} \subseteq X$ (that is open with respect to the metric topology on X) such that $\mathcal{U} = A \cap \mathcal{V}$. In the general topological setting we lack a metric, but this theorem allows us to define subspaces solely in terms of open sets.

Definition 2.1 (**Subspace Topology**) The subspace topology of a subset $A \subseteq X$ with respect to a topological space $(X,\, \tau)$ is the set τ_A defined by:

$$\tau_A = \{\,\mathcal{U} \subseteq A \mid \text{there exists } \mathcal{V} \in \tau \text{ such that } \mathcal{U} = A \cap \mathcal{V}\,\} \tag{9}$$

$\blacksquare$

I am calling this a topology, but *proof by definition* is generally a bad practice. Let's prove the subspace topology is indeed a topology on A.

Theorem 2.1. *If $(X,\, \tau)$ is a topological space, if $A \subseteq X$, and if τ_A is the subspace topology on A, then τ_A is a topology on A.*

Proof. We must prove the four properties of a topology. First, $\emptyset \in \tau_A$ since $\emptyset \in \tau$ and $\emptyset = \emptyset \cap A$. Next, $A \in \tau_A$ since $X \in \tau$ and since $A \subseteq X$ we have

$A = A \cap X$. If $\mathcal{U}, \mathcal{V} \in \tau_A$ then there exists $\tilde{\mathcal{U}}, \tilde{\mathcal{V}} \in \tau$ such that $\mathcal{U} = A \cap \tilde{\mathcal{U}}$ and $\mathcal{V} = A \cap \tilde{\mathcal{V}}$. But then:

$$\mathcal{U} \cap \mathcal{V} = \big(A \cap \tilde{\mathcal{U}}\big) \cap \big(A \cap \tilde{\mathcal{V}}\big) = A \cap \big(\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}}\big) \tag{10}$$

but τ is a topology, so if $\tilde{\mathcal{U}}, \tilde{\mathcal{V}} \in \tau$, then $\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}} \in \tau$. Therefore, $\mathcal{U} \cap \mathcal{V} \in \tau_A$. Lastly, if $\mathcal{O} \subseteq \tau_A$, then for all $\mathcal{U} \in \mathcal{O}$ there is a $\tilde{\mathcal{U}} \in \tau$ such that $\mathcal{U} = A \cap \tilde{\mathcal{U}}$. Let $\tilde{\mathcal{O}}$ be the collection of all such $\tilde{\mathcal{U}}$ for all $\mathcal{U} \in \mathcal{O}$. Then:

$$\begin{aligned} \bigcup \mathcal{O} &= \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} && (11)\\ &= \bigcup_{\tilde{\mathcal{U}} \in \tilde{\mathcal{O}}} \big(A \cap \tilde{\mathcal{U}}\big) && (12)\\ &= A \cap \bigcup_{\tilde{\mathcal{U}} \in \tilde{\mathcal{O}}} \tilde{\mathcal{U}} && (13)\\ &= A \cap \bigcup \tilde{\mathcal{O}} && (14) \end{aligned}$$

But $\tilde{\mathcal{O}} \subseteq \tau$ and τ is a topology, so $\bigcup \tilde{\mathcal{O}} \in \tau$, and hence $\bigcup \mathcal{O} \in \tau_A$. That is, τ_A is a topology on A. □

Example 2.1 The familiar examples of topological subspaces are just metric subspaces. The circle $\mathbb{S}^1$ is a subspace of $\mathbb{R}^2$. The closed interval $[a,\, b] \subseteq \mathbb{R}$ and the open interval $(a,\, b) \subseteq \mathbb{R}$ are subspaces of $\mathbb{R}$ with their respective subspace topologies. The unit sphere $\mathbb{S}^2$ lives in $\mathbb{R}^3$ as a subspace as well. ■

Definition 2.2 (**Inclusion Map**) The inclusion map of a subset $A \subseteq X$ into the set X is the function $\iota : A \to X$ defined by $\iota(x) = x$. ■

Theorem 2.2. *If $(X,\, \tau)$ is a topological space, if $A \subseteq X$, if τ_A is the subspace topology on A, and if $\iota : A \to X$ is the inclusion map, then ι is continuous.*

Proof. Let $\mathcal{V} \in \tau$. Then by the definition of pre-image, $\iota^{-1}[\mathcal{V}]$ is the set of all $x \in A$ such that $\iota(x) \in \mathcal{V}$. But $\iota(x) = x$, so $\iota^{-1}[\mathcal{V}]$ is the set of all $x \in A$ such that $x \in \mathcal{V}$. That is, the set of all elements in $\mathcal{V} \cap A$. But $\mathcal{V} \cap A$ is open in A since $\mathcal{V}$ is open in X. Thus, ι is continuous. □

Theorem 2.3. *If $(X,\, \tau)$ is a topological space, and if τ_A' is a topology on A such that the inclusion map $\iota : A \to X$ is continuous, then $\tau_A \subseteq \tau_A'$ where τ_A is the subspace topology.*

Proof. Let $\mathcal{U} \in \tau_A$. Since τ_A is the subspace topology, there is a $\mathcal{V} \in \tau$ such that $\mathcal{U} = A \cap \mathcal{V}$. But then $\iota^{-1}[\mathcal{V}] = \mathcal{U}$. Since ι is continuous with respect to τ_A' and τ it must be true that $\mathcal{U} \in \tau_A'$. Hence, $\tau_A \subseteq \tau_A'$. □

The subspace topology is the *smallest* topology that makes the inclusion map continuous. This is a way of defining the subspace topology altogether, the intersection of all topologies on A that make ι continuous.

This idea of defining a topology in terms of a function that you want to be continuous is common. There are two directions. If we have a topological space $(X,\,\tau)$, a set Y, and a function $f:X\rightarrow Y$, then the *final topology* on Y with respect to f is the largest topology τ_f that makes f continuous. If X is a set, $(Y,\,\tau)$ is a topological space, and $f:X\rightarrow Y$ is a function, then the *initial topology* is the smallest topology τ_f on X that makes f continuous.

Definition 2.3 (**Final Topology**) The final topology on a set Y with respect to a topological space $(X,\,\tau)$ and a function $f:X\rightarrow Y$ is the set τ_f defined by:

$$\tau_f=\{\,\mathcal{V}\subseteq Y\mid f^{-1}[\mathcal{V}]\in\tau\,\} \tag{15}$$

That is, the set of all subsets of Y whose pre-image is open in X. ■

Again, avoiding proof by definition, the final topology is a topology.

Theorem 2.4. *If $(X,\,\tau)$ is a topological space, if Y is a set, if $f:X\rightarrow Y$ is a function, and if τ_f is the final topology, then τ_f is a topology on Y.*

Proof. First, $\emptyset\in\tau_f$ since $f^{-1}[\emptyset]=\emptyset$ and $\emptyset\in\tau$. Next, $Y\in\tau_f$ since $f^{-1}[Y]=X$ and $X\in\tau$, since τ is a topology. Suppose $\mathcal{U},\mathcal{V}\in\tau_f$. Then:

$$f^{-1}[\mathcal{U}\cap\mathcal{V}]=f^{-1}[\mathcal{U}]\cap f^{-1}[\mathcal{V}] \tag{16}$$

But if $\mathcal{U},\mathcal{V}\in\tau_f$, then $f^{-1}[\mathcal{U}]\in\tau$ and $f^{-1}[\mathcal{V}]\in\tau$. But τ is a topology on X, so $f^{-1}[\mathcal{U}]\cap f^{-1}[\mathcal{V}]\in\tau$. Hence, $\mathcal{U}\cap\mathcal{V}\in\tau_f$. Lastly, let $\mathcal{O}\subseteq\tau_f$. Then for all $\mathcal{U}\in\mathcal{O}$ it is true that $f^{-1}[\mathcal{U}]\in\tau$. But then:

$$f^{-1}\Big[\bigcup\mathcal{O}\Big]=f^{-1}\Big[\bigcup_{\mathcal{U}\in\mathcal{O}}\mathcal{U}\Big] \tag{17}$$

$$=\bigcup_{\mathcal{U}\in\mathcal{O}}f^{-1}[\mathcal{U}] \tag{18}$$

But τ is a topology, so this final union is an element of τ. Hence, $\bigcup\mathcal{O}\in\tau_f$ and τ_f is a topology. □

Theorem 2.5. *If $(X,\,\tau)$ is a topological space, if Y is a set, and if $f:X\rightarrow Y$ is a function, then f is continuous with respect to τ and the final topology τ_f.*

Proof. Given $\mathcal{V}\in\tau_f$, by definition of the final topology we have $f^{-1}[\mathcal{V}]\in\tau$, so f is continuous. □

Theorem 2.6. *If $(X,\,\tau)$ is a topological space, if Y is a set, if $f:X\rightarrow Y$ is a function, if τ_f is the final topology, and if τ_Y is a topology such that f is continuous with respect to τ and τ_Y, then $\tau_Y\subseteq\tau_f$.*

Proof. For let $\mathcal{V}\in\tau_Y$. Since f is continuous with respect to τ and τ_Y it is true that $f^{-1}[\mathcal{V}]\in\tau$. But τ_f is the final topology which is the set of all such $\mathcal{U}\subseteq Y$ such that $f^{-1}[\mathcal{U}]\in\tau$, and therefore $\mathcal{V}\in\tau_f$. That is, $\tau_Y\subseteq\tau_f$. □

This is what was meant by the claim that the final topology is the *largest* topology that makes f continuous.

Definition 2.4 (**Initial Topology**) The initial topology on a set X with respect to a topological space (Y,τ) and a function $f:X\rightarrow Y$ is the set τ_f defined by:

$$\tau_f=\{\,f^{-1}[\mathcal{V}]\mid\mathcal{V}\in\tau\,\} \tag{19}$$

That is, the set of all pre-images of open subsets of Y. ■

The initial topology is, in fact, a topology.

Theorem 2.7. *If X is a set, if (Y,τ) is a topological space, and if $f:X\rightarrow Y$ is a function, then the initial topology τ_f is a topology on X.*

Proof. We have that $\emptyset=f^{-1}[\emptyset]$, so $\emptyset\in\tau_f$. We also have $X=f^{-1}[Y]$, so $X\in\tau_f$. If $\mathcal{U},\mathcal{V}\in\tau_f$, then there are $\tilde{\mathcal{U}},\tilde{\mathcal{V}}\in\tau$ such that $\mathcal{U}=f^{-1}[\tilde{\mathcal{U}}]$ and $\mathcal{V}=f^{-1}[\tilde{\mathcal{V}}]$. But then:

$$\mathcal{U}\cap\mathcal{V}=f^{-1}[\tilde{\mathcal{U}}]\cap f^{-1}[\tilde{\mathcal{V}}]=f^{-1}[\tilde{\mathcal{U}}\cap\tilde{\mathcal{V}}] \tag{20}$$

But τ is a topology, so $\tilde{\mathcal{U}}\cap\tilde{\mathcal{V}}\in\tau$, and hence $\mathcal{U}\cap\mathcal{V}\in\tau_f$. Lastly, if $\mathcal{O}\subseteq\tau_f$, then for all $\mathcal{U}\in\mathcal{O}$ there is a $\tilde{\mathcal{U}}\in\tau$ such that $\mathcal{U}=f^{-1}[\tilde{\mathcal{U}}]$. Let $\tilde{\mathcal{O}}\subseteq\tau$ be the set of all such $\tilde{\mathcal{U}}$. Then:

$$\begin{aligned}
\bigcup\mathcal{O}&=\bigcup_{\mathcal{U}\in\mathcal{O}}\mathcal{U} && (21)\\
&=\bigcup_{\tilde{\mathcal{U}}\in\tilde{\mathcal{O}}}f^{-1}[\tilde{\mathcal{U}}] && (22)\\
&=f^{-1}\Big[\bigcup_{\tilde{\mathcal{U}}\in\tilde{\mathcal{O}}}\tilde{\mathcal{U}}\Big] && (23)\\
&=f^{-1}\Big[\bigcup\tilde{\mathcal{O}}\Big] && (24)
\end{aligned}$$

But τ is a topology so $\bigcup\tilde{\mathcal{O}}\in\tau$ and therefore $\bigcup\mathcal{O}\in\tau_f$. So τ_f is a topology. □

Theorem 2.8. *If X is a set, if (Y,τ) is a topological space, if $f:X\rightarrow Y$ is a function, and if τ_f is the initial topology, then f is continuous.*

Proof. Let $\mathcal{V}\in\tau$. By the definition of the initial topology, $f^{-1}[\mathcal{V}]\in\tau_f$, so f is continuous. □

Theorem 2.9. *If X is a set, if (Y,τ) is a topological space, if $f:X\rightarrow Y$ is a function, if τ_f is the initial topology with respect of f and (Y,τ), and if τ_X is a topology such that f is continuous with respect to τ_X and τ, then $\tau_f\subseteq\tau_X$.*

Proof. Let $\mathcal{U}\in\tau_f$. Then by the definition of the initial topology, there is a $\tilde{\mathcal{U}}\in\tau$ such that $f^{-1}[\tilde{\mathcal{U}}]=\mathcal{U}$. But f is continuous with respect to τ_X and τ, so if $\tilde{\mathcal{U}}\in\tau$, then $f^{-1}[\tilde{\mathcal{U}}]\in\tau_X$. That is, $\mathcal{U}\in\tau_X$ and therefore $\tau_f\subseteq\tau_X$. □

The intial topology is therefore the *smallest* topology that makes f continuous.

Theorem 2.10. *If $(X,\,\tau)$ is a topological space, if $A\subseteq X$, if $\iota:A\rightarrow X$ is the inclusion mapping, and if τ_ι is the initial topology with respect to $(X,\,\tau)$ and ι, then $\tau_\iota=\tau_A$ where τ_A is the subspace topology.*

Proof. τ_A is a topology that makes ι continuous, and hence by the previous theorem, $\tau_\iota\subseteq\tau_A$. Suppose $\mathcal{U}\in\tau_A$. Then there is an open set $\mathcal{V}\in\tau$ such that $\mathcal{U}=\mathcal{V}\cap A$. By the definition of the inclusion mapping, $\iota^{-1}[\mathcal{V}]=\mathcal{V}\cap A$, so $\iota^{-1}[\mathcal{V}]=\mathcal{U}$ and therefore $\mathcal{U}\in\tau_\iota$. That is, $\tau_A\subseteq\tau_\iota$. Therefore, $\tau_A=\tau_\iota$. □

A few pages ago it was stated that homeomorphisms are just relabellings of spaces. With the initial and final topology we can make this precise.

Theorem 2.11. *If $(X,\,\tau)$ is a topological space, if Y is a set, and if $f:X\rightarrow Y$ is a bijection, then there is a unique topology τ_Y on Y such that f is a homeomorphism.*

Proof. Let τ_Y be the final topology from f. This makes f continuous. Given any topology τ_Y' that makes f continuous, since τ_Y is the final topology, we have $\tau_Y'\subseteq\tau_Y$. But then τ_Y is also the initial topology with respect to f^{-1}, making f^{-1} continuous. Then given any topology τ_Y'' that makes f^{-1} continuous, we have $\tau_Y\subseteq\tau_Y''$. Hence any topology τ_Y''' that makes f and f^{-1} continuous must have $\tau_Y\subseteq\tau_Y'''$ and $\tau_Y'''\subseteq\tau_Y$, so $\tau_Y=\tau_Y'''$. That is, τ_Y is the unique topology that makes f a homeomorphism. □

The subspace topology preserves many (but not all) properties of the ambient space.

Theorem 2.12. *If $(X,\,\tau)$ is a Hausdorff topological space, if $A\subseteq X$, and if τ_A is the subspace topology, then $(A,\,\tau_A)$ is a Hausdorff topological space.*

Proof. Let $x,y\in A$ with $x\ne y$. Since $A\subseteq X$ we have $x,y\in X$. But $x\ne y$ and $(X,\,\tau)$ is Hausdorff, so there exist $\mathcal{U},\mathcal{V}\in\tau$ such that $x\in\mathcal{U}$, $y\in\mathcal{V}$, and $\mathcal{U}\cap\mathcal{V}=\emptyset$. Let $\tilde{\mathcal{U}}=A\cap\mathcal{U}$ and $\tilde{\mathcal{V}}=A\cap\mathcal{V}$. Since $x\in A$ and $x\in\mathcal{U}$, we have $x\in A\cap\mathcal{U}$. Since $y\in A$ and $y\in\mathcal{V}$, it is also true that $y\in A\cap\mathcal{V}$. So $x\in\tilde{\mathcal{U}}$ and $y\in\tilde{\mathcal{V}}$. But also:

$$\tilde{\mathcal{U}}\cap\tilde{\mathcal{V}}=\big(A\cap\mathcal{U}\big)\cap\big(A\cap\mathcal{V}\big)=A\cap\big(\mathcal{U}\cap\mathcal{V}\big)=A\cap\emptyset=\emptyset \tag{25}$$

so $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ are open sets such that $x\in\tilde{\mathcal{U}}$, $y\in\tilde{\mathcal{V}}$, and $\tilde{\mathcal{U}}\cap\tilde{\mathcal{V}}=\emptyset$. That is, $(A,\,\tau_A)$ is Hausdorff. □

Theorem 2.13. *If $(X,\,\tau)$ is a second-countable topological space, if $A\subseteq X$, and if τ_A is the subspace topology, then $(A,\,\tau_A)$ is second-countable.*

Proof. Since $(X,\,\tau)$ is second-countable, there is a countable basis $\mathcal{B}\subseteq\tau$. That is, there is a surjection $\mathcal{U}:\mathbb{N}\rightarrow\mathcal{B}$ so that we may list the elements as:

$$\mathcal{B}=\{\,\mathcal{U}_0,\,\dots,\,\mathcal{U}_n,\,\dots\} \tag{26}$$

Let $\tilde{\mathcal{B}} \subseteq \tau_A$ be defined by:

$$\tilde{\mathcal{B}} = \{\, A \cap \mathcal{U}_n \mid n \in \mathbb{N} \,\} \tag{27}$$

$\tilde{\mathcal{B}}$ is countable since the elements are indexed by the natural numbers. We now must show that $\tilde{\mathcal{B}}$ is a basis for $(A,\,\tau_A)$. That is, given $\tilde{\mathcal{U}} \in \tau_A$ we must find $\tilde{\mathcal{O}} \subseteq \tilde{B}$ such that $\bigcup \tilde{\mathcal{O}} = \tilde{\mathcal{U}}$. Since $\tilde{\mathcal{U}} \in \tau_A$, by definition of the subspace topology there is some $\mathcal{V} \in \tau$ such that $\tilde{\mathcal{U}} = A \cap \mathcal{V}$. But $\mathcal{B}$ is a basis for τ so there is $\mathcal{O} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{O} = \mathcal{V}$. Define $\tilde{\mathcal{O}}$ via:

$$\tilde{\mathcal{O}} = \{\, A \cap \mathcal{W} \mid \mathcal{W} \in \mathcal{O} \,\} \tag{28}$$

Then by definition of $\tilde{\mathcal{B}}$ we have $\tilde{\mathcal{O}} \subseteq \tilde{\mathcal{B}}$. Since the elements $\mathcal{W} \in \mathcal{O}$ are subsets of $\mathcal{V}$, and since $\tilde{\mathcal{U}} = A \cap \mathcal{V}$, we have $A \cap \mathcal{W} \subseteq \tilde{\mathcal{U}}$ for all $\mathcal{W} \in \mathcal{O}$, and hence $\bigcup \tilde{\mathcal{O}} \subseteq \tilde{\mathcal{U}}$. Reversing this, let $x \in \tilde{\mathcal{U}}$. Then $x \in A \cap \mathcal{V}$, and hence $x \in \mathcal{V}$. But $\bigcup \mathcal{O} = \mathcal{V}$ so there is some $\mathcal{W} \in \mathcal{O}$ such that $x \in \mathcal{W}$. But then $x \in A$ and $x \in \mathcal{W}$, and hence $x \in A \cap \mathcal{W}$. But $A \cap \mathcal{W} \in \tilde{\mathcal{O}}$, so $x \in \bigcup \tilde{\mathcal{O}}$. Hence, $\tilde{\mathcal{U}} \subseteq \bigcup \tilde{\mathcal{O}}$, meaning $\tilde{\mathcal{U}} = \bigcup \tilde{\mathcal{O}}$. So $\tilde{\mathcal{B}}$ is a countable basis of τ_A and $(A,\,\tau_A)$ is second-countable. □

Theorem 2.14. *If $(X,\,\tau)$ is a first-countable topological space, if $A \subseteq X$, and if τ_A is the subspace topology, then $(A,\,\tau_A)$ is a first-countable topological space.*

Proof. Let $x \in A$. Since $A \subseteq X$ we have that $x \in X$. But $(X,\,\tau)$ is first countable, so there is a countable neighborhood basis $\mathcal{B}$ of x. Since $\mathcal{B}$ is countable, there is a surjection $\mathcal{U} : \mathbb{N} \to \mathcal{B}$ so that we may write the elements as:

$$\mathcal{B} = \{\mathcal{U}_0,\,\dots,\,\mathcal{U}_n,\,\dots\} \tag{29}$$

Define $\tilde{\mathcal{B}}$ as:

$$\tilde{\mathcal{B}} = \{\, A \cap \mathcal{U}_n \mid n \in \mathbb{N} \,\} \tag{30}$$

The set $\tilde{\mathcal{B}}$ is countable. We must now show it is a neighborhood basis of x. First, for all $\mathcal{V} \in \tilde{\mathcal{B}}$ we have $x \in \mathcal{V}$. For if $\mathcal{V} \in \tilde{\mathcal{B}}$ we can write $\mathcal{V} = A \cap \mathcal{U}_n$ for some $n \in \mathbb{N}$. But $\mathcal{B}$ is a neighborhood basis for x and $\mathcal{U}_n \in \mathcal{B}$, so $x \in \mathcal{U}_n$. But $x \in A$, and hence $x \in A \cap \mathcal{U}_n$. So every element of $\tilde{\mathcal{B}}$ contains x. If $\tilde{\mathcal{V}} \in \tau_A$ is such that $x \in \tilde{\mathcal{V}}$, then there is a $\mathcal{V} \in \tau$ such that $\tilde{\mathcal{V}} = A \cap \mathcal{V}$. But $\mathcal{B}$ is a neighborhood basis of x, so there is a $\mathcal{U}_n \in \mathcal{B}$ such that $x \in \mathcal{U}_n$ and $\mathcal{U}_n \subseteq \mathcal{V}$. But then $x \in A \cap \mathcal{U}_n$ and $A \cap \mathcal{U}_n \subseteq A \cap \mathcal{V} = \tilde{\mathcal{V}}$. But $A \cap \mathcal{U}_n$ is an element of $\tilde{\mathcal{B}}$, showing us that $(A,\,\tau_A)$ is first-countable. □

Subspaces of sequential spaces do **not** need to be sequential. Spaces where every subspace is sequential are given a name.

Definition 2.5 (**Fréchet-Urysohn Topological Space**) A Fréchet-Urysohn topological space is a topological space $(X,\,\tau)$ such that for all $A \subseteq X$ it is true that $(A,\,\tau_A)$ is a sequential topological space where τ_A is the subspace topology. ■

Theorem 2.15. *If $(X,\,\tau)$ is a first-countable topological space, then it is a Fréchet-Urysohn topological space.*

Proof. Since first-countable spaces are sequential, and every subspace of a first-countable space is first-countable, every subspace of a first-countable space is also sequential, and hence (X, τ) is a Fréchet-Urysohn space. $\square$

Theorem 2.16. *If (X, τ) is a second-countable topological space, then it is a Fréchet-Urysohn space.*

Proof. Since second-countable spaces are first-countable, this follows from the previous theorem. $\square$

Theorem 2.17. *If (X, τ) is a metrizable topological space, then it is a Fréchet-Urysohn space.*

Proof. Since metrizable spaces are first-countable, this follows from a previous theorem. $\square$

The easiest space to describe that is sequential but *not* Fréchet-Urysohn requires the product topology, which we'll get to soon enough.

Subspaces give rise to the notion of *topological embeddings*, which are another special type of function commonly studied in topology, analysis, and geometry.

Definition 2.6 (**Topological Embedding**) A topological embedding of a topological space (X, τ_X) to a topological space (Y, τ_Y) is a function $f : X \to Y$ such that $f : X \to f[X]$ is a homeomorphism between X and $f[X]$ with respect to the subspace topology $\tau_{Y_{f[X]}}$. $\blacksquare$

Topological embeddings allow us to think of a topological space (X, τ_X) as just a *subspace* of (Y, τ_Y) with the subspace topology.

Theorem 2.18. *If (X, τ) is a topological space, if $A \subseteq X$, if τ_A is the subspace topology, and if ι is the inclusion mapping, then $\iota : A \to X$ is a topological embedding.*

Proof. The image of A is $\iota[A] = A$. $\iota : A \to \iota[A]$ is then just the identity function $\iota : A \to A$ with $\iota(x) = x$, which is a homeomorphism. $\square$