Point-Set Topology: Lecture 16

Ryan Maguire

July 26, 2023

1 Finite Products

Continuing with our trend of building new topological spaces, thus far we have subspaces and quotients. If $(X,\,\tau_X)$ and $(Y,\,\tau_Y)$ are topological spaces, it is possible to put a topology on the Cartesian product $X\times Y$ in a way that respects the topologies τ_X and τ_Y. It is somewhat natural to hope that the set $\tilde{\tau}_{X\times Y}$ defined by:

$$\tilde{\tau}_{X\times Y}=\{\,\mathcal{U}\times\mathcal{V}\subseteq X\times Y\mid\mathcal{U}\in\tau_X\text{ and }\mathcal{V}\in\tau_Y\,\}\tag{1}$$

would be a topology on $X\times Y$, but it usually is not. Consider the real line $\mathbb{R}$ with the standard topology $\tau_{\mathbb{R}}$. This has a basis $\mathcal{B}$ consisting of open intervals of the form $(a,\,b)$ for all $a,b\in\mathbb{R}$. The product $\mathbb{R}\times\mathbb{R}$ should just be the Euclidean plane $\mathbb{R}^2$, but open sets of the form $(a,\,b)\times(c,\,d)$ are *open rectangles*. The product of more general open subsets $\mathcal{U},\mathcal{V}\subseteq\mathbb{R}$ could not possibly form an open disk in the plane, even though we want an open disk to be, well, *open*.

The set $\tilde{\tau}_{X\times Y}$ is nearly a topology. The empty set is contained in it since $\emptyset=\emptyset\times\emptyset$. The entire Cartesian product is an element since $X\in\tau_X$ and $Y\in\tau_Y$, hence $X\times Y\in\tilde{\tau}_{X\times Y}$. It is also closed under intersections since:

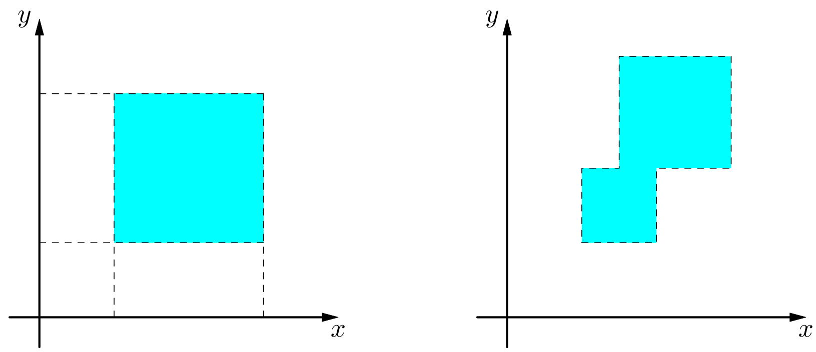
$$\big(\mathcal{U}_0\times\mathcal{V}_0\big)\cap\big(\mathcal{U}_1\times\mathcal{V}_1\big)=\big(\mathcal{U}_0\cap\mathcal{U}_1\big)\times\big(\mathcal{V}_0\cap\mathcal{V}_1\big)\tag{2}$$

and this is an element of $\tilde{\tau}_{X\times Y}$. What fails is the union property. Again, think of $\mathbb{R}\times\mathbb{R}$. The union of two rectangles does not need to be a rectangle. Moreover, open subsets of $\mathbb{R}^2$ such as the open unit disk can not be written in the form $\mathcal{U}\times\mathcal{V}$ for open subsets $\mathcal{U},\mathcal{V}\subseteq\mathbb{R}$. To ensure the product topology is indeed a topology, we need to take the topology *generated* from $\tilde{\tau}_{X\times Y}$.

Definition 1.1 (**Product Topology of Two Topologies**) The product topology of two topologies τ_X and τ_Y on sets X and Y, respectively, is the topology on $X\times Y$ generated by the set $\mathcal{B}$ defined by:

$$\mathcal{B}=\{\,\mathcal{U}\times\mathcal{V}\subseteq X\times Y\mid\mathcal{U}\in\tau_X\text{ and }\mathcal{V}\in\tau_Y\,\}\tag{3}$$

This is denoted $\tau_{X\times Y}$. ■



a: An Open Rectangle in $\mathbb{R}^2$

b: The Union of Rectangles in $\mathbb{R}^2$

Figure 1: Open Subsets of the Plane

Theorem 1.1. *If (X, τ_X) and (Y, τ_Y) are topological spaces, and if $\tau_{X\times Y}$ is the product topology of τ_X and τ_Y, then $(X \times Y, \tau_{X\times Y})$ is a topological space.*

Proof. The product topology $\tau_{X\times Y}$ is a generated topology, by definition, which is hence a topology, so $(X \times Y, \tau_{X\times Y})$ is a topological space. $\square$

The product of two topological spaces better give us the right topologies on familiar spaces, otherwise its useless.

Theorem 1.2. *If $\tau_{\mathbb{R}\times\mathbb{R}}$ is the product topology of $\tau_{\mathbb{R}}$ and $\tau_{\mathbb{R}}$, and if $\tau_{\mathbb{R}^2}$ is the standard Euclidean topology on $\mathbb{R}^2$, then $(\mathbb{R}^2, \tau_{\mathbb{R}^2})$ and $(\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}\times\mathbb{R}})$ are homeomorphic.*

Proof. We simply must prove the topologies $\tau_{\mathbb{R}^2}$ and $\tau_{\mathbb{R}\times\mathbb{R}}$ are the same set, meaning the identity function $\mathrm{id} : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ is a homeomorphism. The standard topology on $\mathbb{R}^2$ is generated by the Euclidean metric:

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||_2 \tag{4}$$

The topology of $\mathbb{R} \times \mathbb{R}$ is generated by open rectangles, which in turn can be generated by open squares, and open squares are the open balls in the max metric:

$$d_{\max}(\mathbf{x}, \mathbf{y}) = \max(|x_0 - y_0|, |x_1 - y_1|) \tag{5}$$

But the Euclidean metric and the max metric are topologically equivalent metrics, meaning they produce the same topologies. So, we're done. $\square$

Definition 1.2 (**Projection Maps**) The projection map of the Cartesian product $X \times Y$ onto X is the function $\mathrm{proj}_X : X \times Y \to X$ defined by $\mathrm{proj}_X\big((x, y)\big) = x$. The projection map of $X \times Y$ onto Y is defined by $\mathrm{proj}_Y : X \times Y \to Y$, $\mathrm{proj}_Y\big((x, y)\big) = y$. $\blacksquare$

Projections are continuous.

Theorem 1.3. *If (X, τ_X) and (Y, τ_Y) are topological spaces, and if $(X \times Y, \tau_{X\times Y})$ is the product space, then $proj_X : X \times Y \to X$ and $proj_Y : X \times Y \to Y$ are continuous.*

Proof. Let $\mathcal{U} \in \tau_X$ and $\mathcal{V} \in \tau_Y$. Then by the definition of the projection map, $\text{proj}_X^{-1}[\mathcal{U}] = \mathcal{U} \times Y$, and $\mathcal{U} \times Y \in \tau_{X\times Y}$, so proj_X is continuous. Similarly, $\text{proj}_Y^{-1}[\mathcal{V}] = X \times \mathcal{V}$, and $X \times \mathcal{V} \in \tau_{X\times Y}$, so proj_Y is continuous. □

Theorem 1.4. *If (X, τ_X) and (Y, τ_Y) are topological spaces, and if $(X \times Y, \tau_{X\times Y})$ is the product space, then $proj_X : X \times Y \to X$ and $proj_X : X \times Y \to Y$ are open maps.*

Proof. Let $\mathcal{W} \in \tau_{X\times Y}$. Since $\tau_{X\times Y}$ is generated by the basis $\mathcal{B}$ defined by:

$$\mathcal{B} = \{\, \mathcal{U} \times \mathcal{V} \mid \mathcal{U} \in \tau_X \text{ and } \mathcal{V} \in \tau_Y \,\} \tag{6}$$

there is some collection $\mathcal{O} \subseteq \mathcal{B}$ such that $\mathcal{W} = \bigcup \mathcal{O}$. But for each $\mathcal{U} \times \mathcal{V} \in \mathcal{O}$ we have $\text{proj}_X[\mathcal{U} \times \mathcal{V}] = \mathcal{U}$ and $\text{proj}_Y[\mathcal{U} \times \mathcal{V}] = \mathcal{V}$. So then $\text{proj}_X[\mathcal{W}]$ is the union of open sets in X, and $\text{proj}_Y[\mathcal{W}]$ is the union of open sets in Y, so $\text{proj}_X[\mathcal{W}] \in \tau_X$ and $\text{proj}_Y[\mathcal{W}] \in \tau_Y$. That is, proj_X and proj_Y are open maps. □

Unlike quotients, which preserve very few properties, products preserve quite a lot of properties.

Theorem 1.5. *If (X, τ_X) and (Y, τ_Y) are Fréchet topological spaces, then $(X \times Y, \tau_{X\times Y})$ is a Fréchet topological space.*

Proof. Let $(x_0, y_0), (x_1, y_1) \in X \times Y$ with $(x_0, y_0) \neq (x_1, y_1)$. Then either $x_0 \neq x_1$ or $y_0 \neq y_1$. Suppose $x_0 \neq x_1$. The proof is symmetric if $y_0 \neq y_1$. Since (X, τ_X) is a Fréchet topological space, and $x_0 \neq x_1$, there exists $\mathcal{U}, \mathcal{V} \in \tau_X$ such that $x_0 \in \mathcal{U}$, $x_0 \notin \mathcal{V}$, $x_1 \in \mathcal{V}$, and $x_1 \notin \mathcal{U}$. But then $\mathcal{U} \times Y$ and $\mathcal{V} \times Y$ are open sets such that $(x_0, y_0) \in \mathcal{U} \times Y$, $(x_0, y_0) \notin \mathcal{V} \times Y$, $(x_1, y_1) \in \mathcal{V} \times Y$, and $(x_1, y_1) \notin \mathcal{U} \times Y$. Hence, $(X \times Y, \tau_{X\times Y})$ is a Fréchet topological space. □

Theorem 1.6. *If (X, τ_X) and (Y, τ_Y) are Hausdorff topological spaces, then $(X \times Y, \tau_{X\times Y})$ is a Hausdorff topological space.*

Proof. Let $(x_0, y_0), (x_1, y_1) \in X \times Y$ with $(x_0, y_0) \neq (x_1, y_1)$. Then either $x_0 \neq x_1$ or $y_0 \neq y_1$. Suppose $x_0 \neq x_1$, the proof is symmetric if $y_0 \neq y_1$. But (X, τ_X) is Hausdorff, so there are opens sets $\mathcal{U}, \mathcal{V} \in \tau_X$ such that $x_0 \in \mathcal{U}$, $x_1 \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. But then $\mathcal{U} \times Y$ and $\mathcal{V} \times Y$ are disjoint open sets such that $(x_0, y_0) \in \mathcal{U} \times Y$ and $(x_1, y_1) \in \mathcal{V} \times Y$. Therefore $(X \times Y, \tau_{X\times Y})$ is a Hausdorff topological space. □

Theorem 1.7. *If (X, τ_X) and (Y, τ_Y) are first-countable topological spaces, then $(X \times Y, \tau_{X\times Y})$ is first-countable.*

Proof. Let $(x, y) \in X \times Y$. Since $x \in X$ and (X, τ_X) is first-countable, there is a countable neighborhood basis $\mathcal{B}_X \subseteq \tau_X$ of x. Since $y \in Y$ and (Y, τ_Y)

is first-countable, there is a countable neighborhood basis $\mathcal{B}_Y \subseteq \tau_Y$ of y. Let $\mathcal{B} \subseteq \tau_{X\times Y}$ be defined as:

$$\mathcal{B} = \{\, \mathcal{U} \times \mathcal{V} \in \tau_{X\times Y} \mid \mathcal{U} \in \mathcal{B}_X \text{ and } \mathcal{V} \in \mathcal{B}_Y \,\} \tag{7}$$

Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are countable, so is $\mathcal{B}$. Since $x \in \mathcal{U}$ for all $\mathcal{U} \in \mathcal{B}_X$ and $y \in \mathcal{V}$ for all $\mathcal{V} \in \mathcal{B}_Y$, we have that $(x,\, y) \in \mathcal{U} \times \mathcal{V}$ for all $\mathcal{U} \times \mathcal{V} \in \mathcal{B}$. We now need to show that $\mathcal{B}$ is a neighborhood basis of $(x,\, y)$. Let $\mathcal{W} \in \tau_{X\times Y}$ be a set containing $(x,\, y)$. Then, by the definition of the product topology, there is a subset $\mathcal{O} \subseteq \tau_{X\times Y}$ such that $\mathcal{W} = \bigcup \mathcal{O}$ and all elements of $\mathcal{O}$ are of the form $\mathcal{U} \times \mathcal{V}$ with $\mathcal{U} \in \tau_X$ and $\mathcal{V} \in \tau_Y$. But $(x,\, y) \in \mathcal{W}$, so there is an element $\mathcal{U} \times \mathcal{V} \in \mathcal{O}$ with $(x,\, y) \in \mathcal{U} \times \mathcal{V}$. Then $x \in \mathcal{U}$ and $y \in \mathcal{V}$. But $\mathcal{B}_X$ is a neighborhood basis of x, so there is a $\tilde{\mathcal{U}} \in \mathcal{B}_X$ such that $\tilde{\mathcal{U}} \subseteq \mathcal{U}$. But $\mathcal{B}_Y$ is a neighborhood basis of y so there is a $\tilde{\mathcal{V}} \in \mathcal{B}_Y$ such that $\tilde{\mathcal{V}} \subseteq \mathcal{V}$. But then $\tilde{\mathcal{U}} \times \tilde{\mathcal{V}}$ is an element of $\mathcal{B}$ such that $\tilde{\mathcal{U}} \times \tilde{\mathcal{V}} \subseteq \mathcal{U} \times \mathcal{V}$, and since $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{W}$, we have $\tilde{\mathcal{U}} \times \tilde{\mathcal{V}} \subseteq \mathcal{W}$. Hence, $\mathcal{B}$ is a countable neighborhood basis of $(x,\, y)$ and $(X \times Y,\, \tau_{X\times Y})$ is first-countable. □

Theorem 1.8. *If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are second-countable topological spaces, then $(X \times Y,\, \tau_{X\times Y})$ is second-countable.*

Proof. Since $(X,\, \tau_X)$ is second-countable, there is a countable basis $\mathcal{B}_X$ for τ_X. Since $(Y,\, \tau_Y)$ is second-countable, there is a countable basis $\mathcal{B}_Y$ for τ_Y. Let $\mathcal{B}$ be defined by:

$$\mathcal{B} = \{\, \mathcal{U} \times \mathcal{V} \mid \mathcal{U} \in \mathcal{B}_X \text{ and } \mathcal{V} \in \mathcal{B}_Y \,\} \tag{8}$$

Then $\mathcal{B}$ has a cardinality bounded by $\mathbb{N} \times \mathbb{N}$, which is countable. We now need to prove $\mathcal{B}$ is a countable basis of $\tau_{X\times Y}$. To do this it suffices to show that any open set $\mathcal{W} \in \tau_{X\times Y}$ can be written as $\mathcal{W} = \bigcup \mathcal{O}$ for some set $\mathcal{O} \subseteq \mathcal{B}$. Define $\mathcal{O}$ via:

$$\mathcal{O} = \{\, \mathcal{U} \times \mathcal{V} \in \mathcal{B} \mid \mathcal{U} \times \mathcal{V} \subseteq \mathcal{W} \,\} \tag{9}$$

Then by definition of $\mathcal{O}$ we have that $\bigcup \mathcal{O} \subseteq \mathcal{W}$. Let's reverse this. Let $(x,\, y) \in \mathcal{W}$. Then, since projection maps are open maps, $x \in \mathrm{proj}_X(\mathcal{W})$, which is an open set. Since $\mathcal{B}_X$ is a basis there is some $\mathcal{U} \in \mathcal{B}_X$ such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq \mathrm{proj}_X(\mathcal{W})$. Similarly there is some $\mathcal{V} \in \mathcal{B}_Y$ such that $y \in \mathcal{V}$ and $\mathcal{V} \subseteq \mathrm{proj}_Y(\mathcal{W})$. But then $(x,\, y) \in \mathcal{U} \times \mathcal{V}$ and $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{W}$. Morever, since $\mathcal{U} \in \mathcal{B}_X$ and $\mathcal{V} \in \mathcal{B}_Y$, we have that $\mathcal{U} \times \mathcal{V} \in \mathcal{B}$. But then $\mathcal{U} \times \mathcal{V} \in \mathcal{O}$, and hence $(x,\, y) \in \bigcup \mathcal{O}$. That is, $\mathcal{W} \subseteq \bigcup \mathcal{O}$, and therefore $\mathcal{W} = \bigcup \mathcal{O}$. So $\mathcal{B}$ is a countable basis. □

The way to often think of product spaces $(X \times Y,\, \tau_{X\times Y})$ is to take a copy of X and attach it to each $y \in Y$. Similarly, you could think of attaching a copy of Y to each $x \in X$. This mode of thinking makes it easier to visualize certain product spaces.

Example 1.1 The plane $\mathbb{R}^2$ can be thought of as attaching a copy of $\mathbb{R}$ to each real number $x \in \mathbb{R}$. That is, think of $\mathbb{R}$ as a horizontal line, and at each $x \in \mathbb{R}$ attach a copy of $\mathbb{R}$ that is directed vertically. The result is $\mathbb{R}^2$. ■

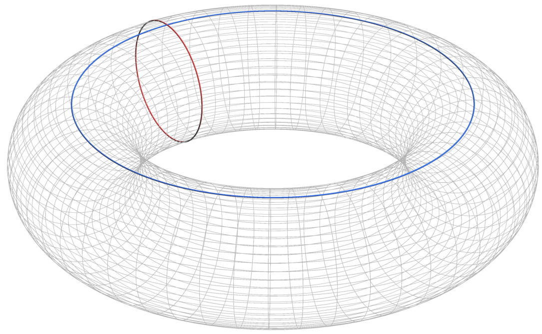


Figure 2: The Torus $\mathbb{T}^2$

Example 1.2 Letting $\mathbb{S}^1$ denote the unit circle in the plane, $\mathbb{R}\times\mathbb{S}^1$ is a *cylinder*. At every point $\mathbf{x}\in\mathbb{S}^1$, attach a copy of the real line that is directed upwards out of the plane. The result is a cylinder in $\mathbb{R}^3$. ■

Example 1.3 The space $\mathbb{S}^1\times\mathbb{S}^1$ is the torus, and this is denoted $\mathbb{T}^2$. As a set it lives as a subset of $\mathbb{R}^4$ since $\mathbb{S}^1\subseteq\mathbb{R}^2$, and hence $\mathbb{S}^1\times\mathbb{S}^1\subseteq\mathbb{R}^4$. However, it is far easier to visualize this space as a subset of $\mathbb{R}^3$ using our intuition of *attaching* spaces to points. At every point on the first circle $\mathbb{S}^1$ we attach a copy of the second circle $\mathbb{S}^1$. The result is a *circle of circles*, which is the torus, and this can be embedded into $\mathbb{R}^3$. This is done in Fig. 2. ■

Products also preserve the property of being metrizable. We've seen this before when we constructed several equivalent metrics on $\mathbb{R}^2$ using the standard metric on $\mathbb{R}$.

Theorem 1.9. *If $(X,\,\tau_X)$ and $(Y,\,\tau_Y)$ are metrizable topological spaces, then $(X\times Y,\,\tau_{X\times Y})$ is metrizable.*

Proof. Since $(X,\,\tau_X)$ is metrizable, there is a metric d_X on X that induces τ_X. Since $(Y,\,\tau_Y)$ is metrizable, there is a metric d_Y on Y that induces τ_Y. Let $d_{X\times Y}$ be defined by:

$$d_{X\times Y}\big((x_0,\,y_0),\,(x_1,\,y_1)\big)=d_X(x_0,\,x_1)+d_Y(y_0,\,y_1) \tag{10}$$

Then $d_{X\times Y}$ is a metric. It is positive-definite, symmetric, and satisfies the triangle inequality since d_X and d_Y do. It also induces $\tau_{X\times Y}$. A set is open with the metric $d_{X\times Y}$ if and only if it is the union of open balls with the $d_{X\times Y}$ metric. Since sets of the form $\mathcal{U}\times\mathcal{V}\in\tau_{X\times Y}$ with $\mathcal{U}\in\tau_X$ and $\mathcal{V}\in\tau_Y$ form a basis for $\tau_{X\times Y}$, it suffices to show that all $\mathcal{U}\times\mathcal{V}$ are the union of open balls. For all $(x,\,y)\in\mathcal{U}\times\mathcal{V}$, since d_X induces τ_X, there is an $r_x>0$ such that $B_{r_x}^{(X,\,d_X)}(x)\subseteq\mathcal{U}$. Since d_Y induces τ_Y, there is an $r_y>0$ such that $B_{r_y}^{(Y,\,d_Y)}(y)\subseteq\mathcal{V}$. Let $r_{(x,\,y)}=\min(r_x,\,r_y)$. Then $B_{r_{(x,\,y)}}^{(X\times Y,\,d_{X\times Y})}\big((x,\,y)\big)$ is a subset of $\mathcal{U}\times\mathcal{V}$. For let $(a,\,b)\in B_{r_{(x,\,y)}}^{(X\times Y,\,d_{X\times Y})}\big((x,\,y)\big)$. But then:

$$d_X(a,\,x)\le d_X(a,\,x)+d_Y(b,\,y)=d_{X\times Y}\big((a,\,x),\,(b,\,y)\big)<r_{(x,\,y)}\le r_x \tag{11}$$

so $a \in B_{r_x}^{(X,\, d_X)}(x)$, and hence $a \in \mathcal{U}$. Similarly:

$$d_Y(b,\, y) \leq d_X(a,\, x) + d_Y(b,\, y) = d_{X\times Y}\big((a,\, x),\, (b,\, y)\big) < r_{(x,\, y)} \leq r_y \tag{12}$$

so $b \in B_{r_y}^{(Y,\, d_Y)}(y)$, and hence $b \in \mathcal{V}$. So $(a,\, b) \in \mathcal{U} \times \mathcal{V}$. Let $\mathcal{W}_{(x,\, y)}$ be the $r_{(x,\, y)}$ ball in $(X \times Y,\, d_{X\times Y})$ centered at $(x,\, y)$ for all $(x,\, y) \in \mathcal{U} \times \mathcal{V}$. Then, since $(x,\, y) \in \mathcal{W}_{(x,\, y)}$, we have:

$$\mathcal{U} \times \mathcal{V} \subseteq \bigcup_{(x,\, y)\in\mathcal{U}\times\mathcal{V}} \mathcal{W}_{(x,\, y)} \tag{13}$$

But also $\mathcal{W}_{(x,\, y)} \subseteq \mathcal{U} \times \mathcal{V}$ for all $(x,\, y) \in \mathcal{U} \times \mathcal{V}$, so:

$$\bigcup_{(x,\, y)\in\mathcal{U}\times\mathcal{V}} \mathcal{W}_{(x,\, y)} \subseteq \mathcal{U} \times \mathcal{V} \tag{14}$$

Hence $\mathcal{U} \times \mathcal{V}$ is the union of open balls in the $d_{X\times Y}$ metric. That is, an open set in $\tau_{X\times Y}$ is the union of open balls, meaning $\tau_{X\times Y} \subseteq \tau_{d_{X\times Y}}$. We must reverse this. This is, we must show that the union of open balls is open with respect to $\tau_{X\times Y}$. To do this it suffices to show that open balls with the $d_{X\times Y}$ metric are open in the topology $\tau_{X\times Y}$, since then the union of open balls would be the union of open sets, which is therefore open. So let $(x,\, y) \in X \times Y$ and $r > 0$. Let $(a,\, b) \in B_r^{(X\times Y,\, d_{X\times Y})}\big((x,\, y)\big)$. Let $r_{(a,\, b)} = \frac{1}{2}\big(r - d_{X\times Y}\big((a,\, x),\, (b,\, y)\big)$. Let $\mathcal{U}_a = B_{r_{(a,\, b)}}^{(X,\, d_X)}(x)$ and $\mathcal{V}_b = B_{r_{(a,\, b)}}^{(Y,\, d_Y)}(y)$. Then $\mathcal{U}_a \times \mathcal{V}_b \subseteq B_r^{(X\times Y,\, d_{X\times Y})}\big((x,\, y)\big)$. For if $(x_0,\, y_0) \in \mathcal{U}_a \times \mathcal{V}_b$, then:

$$\begin{aligned}
&d_{X\times Y}\big((x_0,\, y_0),\, (x,\, y)\big)\\
&\quad\leq d_{X\times Y}\big((x_0,\, y_0),\, (a,\, b)\big) + d_{X\times Y}\big((a,\, b),\, (x,\, y)\big) && (15)\\
&\quad= d_X(x_0,\, a) + d_Y(y_0,\, b) + d_{X\times Y}\big((a,\, b),\, (x,\, y)\big) && (16)\\
&\quad< \frac{1}{2}\Big(r - d_{X\times Y}\big((a,\, b),\, (x,\, y)\big)\Big) + \frac{1}{2}\Big(r - d_{X\times Y}\big((a,\, b),\, (x,\, y)\big)\Big)\\
&\qquad + d_{X\times Y}\big((a,\, b),\, (x,\, y)\big) && (17)\\
&\quad= r && (18)
\end{aligned}$$

So $B_r^{(X\times Y,\, \tau_{X\times Y})}\big((x,\, y)\big)$ can be written as the union of all such $\mathcal{U}_a \times \mathcal{V}_b$ for all $(a,\, b)$ in the set, meaning open balls with the $d_{X\times Y}$ metric are open. So a set is open in $\tau_{X\times Y}$ if and only if it is open with respect to $d_{X\times Y}$, so $d_{X\times Y}$ induces the topology and $(X \times Y,\, \tau_{X\times Y})$ is metrizable. □

Products can be performed for any finite collection of topological spaces. We replace $X \times Y$ with $\prod_{n\in\mathbb{Z}_N} X_n$, given a collection of $N \in \mathbb{N}$ topological spaces $(X_n,\, \tau_n)$. The topology is generated by sets of the form $\prod_{n\in\mathbb{Z}_N} \mathcal{U}_n$ where $\mathcal{U}_n \in \tau_n$ for all $n \in \mathbb{Z}_N$. All of the previous theorems still hold for finite products, and the proofs are done by induction. (Try it yourself, I can't prove everything for you!)

- The finite product of Fréchet spaces is Fréchet.
- The finite product of Hausdorff spaces is Hausdorff.
- The finite product of first-countable spaces is first-countable.
- The finite product of second-countable spaces is second-countable.
- The finite product of metrizable spaces is metrizable.

The product of sequential spaces does **not** need to be sequential.

2 Infinite Products

When we go from the finite world to the infinite things get a bit problematic. First, how do we even topologize an infinite product? There are two ways: the *obvious* way, and the correct one. It took me a long time to realize that the obvious way is not the correct one. I've a few examples up my sleeves, so hopefully you'll realize sooner than I did. The obvious way is the *box topology.* Given a set I such that for all $\alpha \in I$ we have that $(X_\alpha,\,\tau_\alpha)$ is a topological space, we can form the following basis $\mathcal{B}_{\text{Box}}$ for the product:

$$\mathcal{B}_{\text{Box}} = \Big\{\prod_{\alpha\in I}\mathcal{U}_\alpha \mid \mathcal{U}_\alpha \in \tau_\alpha \text{ for all } \alpha \in I\Big\} \tag{19}$$

This should definitely be considered the obvious way. We stole our idea for finite products and just generated a topology using this. This idea is horrible, unfortunately. The set $\mathbb{R}^\infty = \prod_{n=0}^{\infty}\mathbb{R}$ is the set of all sequences in $\mathbb{R}$. The function $f:\mathbb{R}\rightarrow\mathbb{R}^\infty$ defined by $f(x) = a:\mathbb{N}\rightarrow\mathbb{R}$ where $a_n = x$ for all $n\in\mathbb{N}$ certainly seems like a simple enough function. Intuitively, this is the function:

$$f(x) = (x,\,x,\,x,\,\dots,\,x,\,\dots) \tag{20}$$

Note that in each component the function is indeed continuous. That is, $f_n:\mathbb{R}\rightarrow\mathbb{R}$ defined by $f_n(x) = f(x)_n$ is just $f_n(x) = x$, which is continuous. With respect to the box topology, f is *nowhere continuous.* Talk about aweful! This is one of the simplest functions one could describe from $\mathbb{R}$ to $\mathbb{R}^\infty$ and yet the box topology says it's everywhere discontinuous.

If you were given a function $f:\mathbb{R}\rightarrow\mathbb{R}^3$ from calculus like:

$$f(t) = \big(t^2+1,\,\sin(t)e^t,\,t^3-t\big) \tag{21}$$

would you bother checking that the pre-image of an open set is open to determine f is continuous? Of course not, you'd note that in the x coordinate we have $x(t) = t^2+1$, which is a polynomial, so it is continuous. In the y coordinate you have $y(t) = \sin(t)e^t$, the product of continuous functions, so continuous. In the z coordinate you have $z(t) = t^3-t$, another polynomial. Since f is continuous in all of its components, you'd rightly conclude that f is continuous. This is the way continuous functions should work with infinite products as well, but the box topology lacks such a feature. The problem is the box topology is way to big. We need to restrict which sets we consider open if we want a nice topology on the product. Let's try the following. Define $\mathcal{B}_{\text{Prod}}$ as:

$$\mathcal{B}_{\text{Prod}} = \Big\{ \prod_{\alpha\in I} \mathcal{U}_\alpha \mid \mathcal{U}_\alpha \in \tau_\alpha \text{ and } \mathcal{U}_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in I \Big\} \tag{22}$$

Let τ_{Box} and τ_{Prod} be the topologies generated from $\mathcal{B}_{\text{Box}}$ and $\mathcal{B}_{\text{Prod}}$, respectively. Hopefully from the definition it is clear that $\tau_{\text{Prod}} \subseteq \tau_{\text{Box}}$. The product topology is formed in a similar manner to the box topology, but with a major restriction on which sets we use to generate our topology.

The product topology is precisely the topology that makes it so that a function $f : Y \to \prod_{\alpha\in I} X_\alpha$, with respect to a topological space $(Y,\,\tau_Y)$, is continuous if and only if $f_\alpha : Y \to X_\alpha$, the component function, is continuous for all $\alpha \in I$. You will prove this in your homework.

Note that the box topology and the product topology are the same for finite products. It's only in the infinite world where things differ. The product topology also has the following nice feature.

Theorem 2.1. *If $\mathcal{X}$ is a countable set of metrizable spaces $(X_n,\,\tau_n)$, if $\tau_\prod$ is the product topology on $\prod_{n\in\mathbb{N}} X_n$, then $(\prod_{n\in\mathbb{N}} X_n,\,\tau_\prod)$ is metrizable.*

Proof. For each space $(X_n,\,\tau_n)$ there is a metric d_n that induces the topology. These metrics may be unbounded, so define ρ_n to be the topologically equivalence metric given by:

$$\rho_n(x,\,y) = \frac{d_n(x,\,y)}{1 + d_n(x,\,y)} \tag{23}$$

Define $d_\prod$ by:

$$d_\prod(a,\,b) = \sum_{n=0}^{\infty} \frac{\rho_n(a_n,\,b_n)}{2^n} \tag{24}$$

(Remember, $a \in \prod_{n\in\mathbb{N}} X_n$ is a sequence $a : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} X_n$ such that $a_n \in X_n$ for all $n \in \mathbb{N}$). This sum converges since each ρ_n is bounded by 1, so we have a valid function on $\prod_{n\in\mathbb{N}} X_n$. It is also metric. It is positive-definite, symmetric, and satisfies the triangle inequality since all of the ρ_n do. The product topology has a subbasis of open sets of the form:

$$\tilde{\mathcal{U}} = \prod_{n=0}^{\infty} \mathcal{U}_n \tag{25}$$

where $\mathcal{U}_n \in \tau_n$ for all $n \in \mathbb{N}$, and $\mathcal{U}_n = X_n$ for all but **one** $n \in \mathbb{N}$ (this is a subbasis, not a basis). So we need to just show that these subbasis elements are open with respect to $d_{\prod}$. Given $a \in \tilde{\mathcal{U}}$, $a_n \in \mathcal{U}_n \in \tau_n$, since ρ_n induces τ_n, there is an $r' > 0$ such that $B_r^{(X_n,\, \rho_n)}(a_n) \subseteq \mathcal{U}_n$. Let $r = r'/2^n$. But then $B_r^{(\prod_n X_n,\, d_{\prod})}(a) \subseteq \tilde{\mathcal{U}}$ since given $b \in B_r^{(\prod_n X_n,\, d_{\prod})}(a)$, we have:

$$\frac{1}{2^n}\rho_n(a_n,\, b_n) \leq \sum_{k=0}^{\infty} \frac{\rho_k(a_k,\, b_k)}{2^k} < r = \frac{r'}{2^n} \tag{26}$$

and hence $\rho_n(a_n,\, b_n) < r'$, so $b_n \in \mathcal{U}_n$, and therefore $b \in \tilde{\mathcal{U}}$. Next, to show open balls are open. Let $r > 0$ and a an element of the product set and choose $N \in \mathbb{N}$ such that $1/2^N < r/2$. Let $\mathcal{U}_n$ be the $r/4$ ball centered at a_n for all $n \in \mathbb{Z}_N$, and $\mathcal{U}_n = X_n$ for all $n \geq N$. Then $\prod_{n\in\mathbb{N}} \mathcal{U}_n \in \tau_{\prod}$ by the definition of the product topology. But also $\prod_{n\in\mathbb{N}} \mathcal{U}_n \subseteq B_r^{(\prod_n X_n,\, d_{\prod})}(a)$. For if $b \in \prod_{n\in\mathbb{N}} \mathcal{U}_n$, then:

$$d_{\prod}(a,\, b) = \sum_{n=0}^{\infty} \frac{\rho_n(a_n,\, b_n)}{2^n} \tag{27}$$

$$= \sum_{n=0}^{N-1} \frac{\rho_n(a_n,\, b_n)}{2^n} + \sum_{n=N}^{\infty} \frac{\rho_n(a_n,\, b_n)}{2^n} \tag{28}$$

$$< \sum_{n=0}^{N-1} \frac{r}{4}\frac{1}{2^n} + \sum_{n=N}^{\infty} \frac{1}{2^n} \tag{29}$$

$$< \frac{r}{2} + \frac{r}{2} \tag{30}$$

$$= r \tag{31}$$

So we can find an open set containing a that fits entirely inside of the r ball centered at a. This can be modified for all elements of the r ball centered at a, meaning this set can be written as the union of open sets, which is therefore open. So open balls with respect to $d_{\prod}$ are open in $\tau_{\prod}$, and open sets in $\tau_{\prod}$ are open with respect to $d_{\prod}$. Hence $d_{\prod}$ induces the topology and $(\prod_{n\in\mathbb{N}} X_n,\, \tau_{\prod})$ is metrizable. $\square$

The claim is not true for uncountable products. The product of uncountably many metrizable spaces need not be first-countable, and hence cannot possibly be metrizable.

3 Homotopy and Homotopy Equivalence

Homeomorphism is the main notion of *sameness* for topological spaces. If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are homeomorphic, then topologically they are indistinguishable and may as well be regarded as the same topological space. There is another notion of *same* that is far weaker, but also very intuitive and pictorial. This idea is described by *homotopies*. Homotopy is motivated by curves in the plane. Suppose we have $f : [0,\, 1] \to \mathbb{R}$ and $g : [0,\, 1] \to \mathbb{R}$ defined by $f(x) = \exp(x)$ and $g(x) = x^3$. We visualize these functions as curves

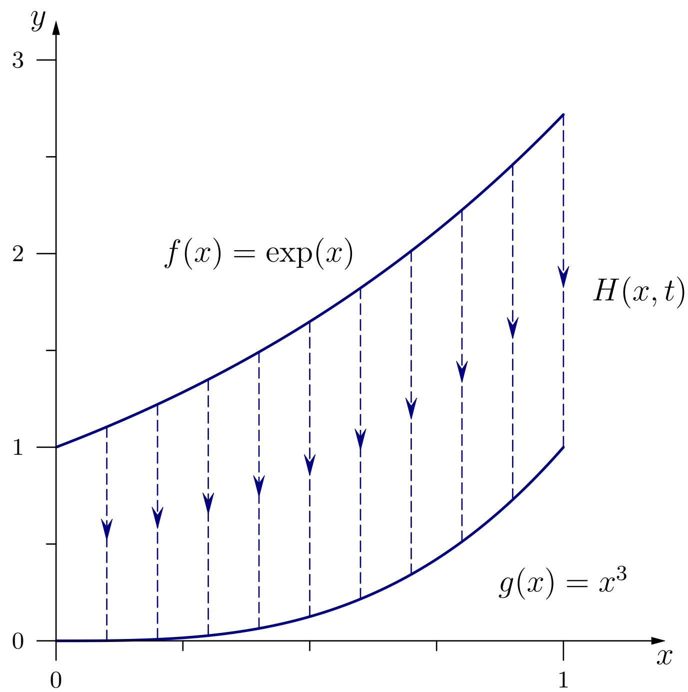


Figure 3: Homotopy Between Curves

$\alpha, \beta : [0,\, 1] \to \mathbb{R}^2$ defined by $\alpha(t) = \big(t,\, f(t)\big)$ and $\beta(t) = \big(t,\, g(t)\big)$. A homotopy from the curve α to the curve β is a way of *continuously* deforming α into β. In the plane this can be done by dragging the point $\alpha(t)$ to the point $\beta(t)$ along the straight line between them for all $t \in [0,\, 1]$. This is shown in Fig. 3. We use this to motivate homotopies in general. It should be a way of continuously deforming one function into another.

Definition 3.1 (**Homotopy**) A homotopy from a continuous function $f_0 : X \to Y$ to a continuous function $f_1 : X \to Y$ between topological spaces $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ is a continuous function $H : X \times [0,\, 1] \to Y$, where $[0,\, 1]$ has the subspace topology from $\mathbb{R}$ and $X \times [0,\, 1]$ has the product topology, such that for all $x \in X$ we have $H(x,\, 0) = f_0(x)$ and $H(x,\, 1) = f_1(x)$. ■

Some spaces, such as Euclidean spaces, are *too nice* and have the property that all continuous functions are homotopic to one another.

Theorem 3.1. *If $(X,\, \tau)$ is a topological space, if $\tau_{\mathbb{R}^n}$ is the standard Euclidean topology on $\mathbb{R}^n$, and if $f_0, f_1 : X \to \mathbb{R}^n$ are continuous functions, then there is a homotopy $H : X \times [0,\, 1] \to \mathbb{R}^n$ between f_0 and f_1.*

Proof. Define $H : X \times [0,\, 1] \to \mathbb{R}^n$ via:

$$H(x,\, t) = (1 - t)f_0(x) + tf_1(x) \tag{32}$$

Since multiplication and addition is continuous in $\mathbb{R}^n$, and since f_0 and f_1 are continuous, H is continuous. Moreover, for all $x \in X$ we have $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$, so H is a homotopy between f_0 and f_1. □

Homotopic is an equivalence relation on the set of all continuous functions between (X, τ_X) and (Y, τ_Y).

Theorem 3.2. *If (X, τ_X) and (Y, τ_Y) are topological spaces, and if $f : X \to Y$ is continuous, then f is homotopic to itself.*

Proof. Let $H : X \times [0, 1] \to Y$ be defined by $H(x, t) = f(x)$. Then, since f is continuous, so is H. However $H(x, 0) = f(x)$ and $H(x, 1) = f(x)$, so H is a homotopy from f to itself. □

Theorem 3.3. *If (X, τ_X) and (Y, τ_Y) are topological spaces, and if $f_0, f_1 : X \to Y$ are continuous functions such that f_0 is homotopic to f_1, then f_1 is homotopic to f_0.*

Proof. Since f_0 is homotopic to f_1 there is a homotopy $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in X$. Define $G : X \times [0, 1] \to Y$ via:

$$G(x, t) = H(x, 1 - t) \tag{33}$$

Since $h : [0, 1] \to [0, 1]$ defined by $h(t) = 1 - t$ is continuous, and since H is continuous, G is continuous as well. But $G(x, 0) = H(x, 1) = f_1(x)$ and $G(x, 1) = H(x, 0) = f_0(x)$. So G is a homotopy from f_1 to f_0. □

Transitivity requires the pasting lemma, a fundamental result about building continuous functions by gluing two functions together.

Theorem 3.4 (The Pasting Lemma). *If (X, τ_X) and (Y, τ_Y) are topological spaces, if $A, B \subseteq X$ are closed subsets, if $X = A \cup B$, and if $f_0 : A \to Y$ and $f_1 : B \to Y$ are continuous functions with the subspace topologies on A and B such that for all $x \in A \cap B$ it is true that $f_0(x) = f_1(x)$, then the function $f : X \to Y$ defined by:*

$$f(x) = \begin{cases} f_0(x) & x \in A \\ f_1(x) & x \in B \end{cases} \tag{34}$$

is continuous.

Proof. First, f is a function. It is well-defined since on $A \cap B$ the functions f_0 and f_1 agree. Second, for all $x \in X$ there is a $y \in Y$ such that $f(x) = y$ since $A \cup B = X$, so both A and B cover X. Now to show it is continuous. Let $\mathcal{D} \subseteq Y$ be closed. Then since f_0 is continuous, $f_0^{-1}[\mathcal{D}]$ is closed. But f_1 is also continuous, so $f_1^{-1}[\mathcal{D}]$ is closed. But then:

$$f^{-1}[\mathcal{D}] = f_0^{-1}[\mathcal{D}] \cup f_1^{-1}[\mathcal{D}] \tag{35}$$

Hence $f^{-1}[\mathcal{D}]$ is the union of two closed sets, which is closed. Therefore, f is continuous. □

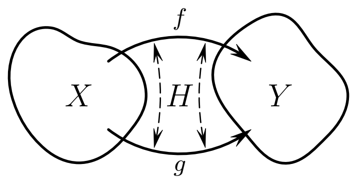


Figure 4: Homotopy Between Continuous Functions

Theorem 3.5. *If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are topological spaces, if $f_0, f_1, f_2 : X \to Y$ are continuous, if f_0 is homotopic to f_1, and if f_1 is homotopic to f_2, then f_0 is homotopic to f_2.*

Proof. Since f_0 is homotopic to f_1, there is a homotopy $H : X \times [0,\, 1] \to Y$ such that $H(x,\, 0) = f_0(x)$ and $H(x,\, 1) = f_1(x)$. Since f_1 and f_2 are homotopic, there is a homotopy $G : X \times [0,\, 1] \to Y$ such that $G(x,\, 0) = f_1(x)$ and $G(x,\, 1) = f_2(x)$. Define $F : X \times [0,\, 1] \to Y$ via:

$$F(x,\, t) = \begin{cases} H(x,\, 2t) & 0 \le t \le \frac{1}{2} \\ G(x,\, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \tag{36}$$

This is well-defined since $F(t,\, \frac{1}{2}) = H(x,\, 1) = G(x,\, 0) = f_1(x)$ for all $x \in X$. It is also continuous by the pasting lemma, since both H and G are continuous. But also $F(x,\, 0) = H(x,\, 0) = f_0(x)$ and $F(x,\, 1) = G(x,\, 1) = f_2(x)$, so F is a homotopy between f_0 and f_2. □

For the more general picture with $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ being arbitrary topological spaces, we use Fig. 4 for guiding intuition. In the case $(X,\, \tau_X) = ([0,\, 1],\, \tau_{[0,\, 1]})$, the closed unit interval with the subspace topology, we again think of curves in the space $(Y,\, \tau_Y)$. See Fig. 5

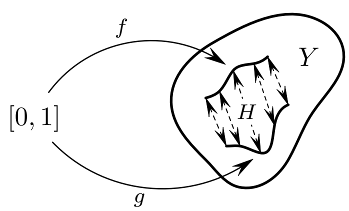


Figure 5: Homotopy Between Curves

Think of the circle $\mathbb{S}^1$ with the subspace topology from $\mathbb{R}^2$. Given two continuous functions $f_0, f_1 : \mathbb{S}^1 \to \mathbb{S}^1$, do you think it must be true that f_0 and f_1 are homotopic, like was the case with $\mathbb{R}^n$? Let's alter the question slightly. Consider the functions $f_0, f_1 : [0,\, 1] \to \mathbb{S}^1$ defined by $f_0(t) = \big(\cos(\pi t),\, \sin(\pi t)\big)$ and $f_1(t) = \big(\cos(\pi t),\, -\sin(\pi(t))\big)$. These functions start and end at the same points on the circle. Can you deform f_0 into f_1 while keeping the endpoints fixed and staying inside the circle? If you could leave the circle, the problem would be

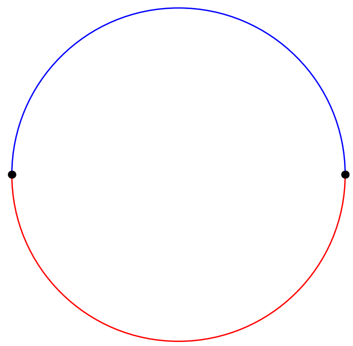


Figure 6: Two Curves on $\mathbb{S}^1$

easy, just do the straight line homotopy $H(s, t) = (1-t)f_0(s) + tf_1(s)$, but that is not the question. You may not change the endpoints and you can't leave the circle. Hopefully this seems impossible, and because this is impossible it is not true that all functions $f_0, f_1 : \mathbb{S}^1 \to \mathbb{S}^1$ are homotopic.

The feature Euclidean space has is that it is *contractible*, it can be shrunk down continuously to a point. The circle has a large hole in it and cannot be collapsed to a point. To make this precise, now is the time to talk about *homotopy equivalences*. First, one more definition.

Definition 3.2 (**Homotopy Inverse**) A homotopy inverse for a continuous function $f : X \to Y$ from a topological space (X, τ_X) to a topological space (Y, τ_Y) is a continuous function $g : Y \to X$ such that $g \circ f : X \to X$ is homotopic to the identity function id_X and $f \circ g : Y \to Y$ is homotopic to the identity function id_Y. ■

Definition 3.3 (**Homotopy Equivalence Topological Spaces**) Homotopy equivalent topological spaces are topological spaces (X, τ_X) and (Y, τ_Y) such that there is a continuous function $f : X \to Y$ that has a homotopy inverse $g : Y \to X$. f and g are called *homotopy equivalences*. ■

Homotopy equivalent is a new notion of *sameness* for topological spaces, but it is far weaker than homeomorphic. It is also extremely visual and intuitive, once you get the idea. In homeomorphisms you are allowed to continuously and bijectively move your space around. With homotopy equivalence you are allowed to do a lot more. You can squeeze points together, stretch points out, you just can't tear your space. Homeomorphisms are, in particular, homotopy equivalences.

Theorem 3.6. *If (X, τ_X) and (Y, τ_Y) are homeomorphic topological spaces, then they are homotopy equivalent.*

Proof. Since (X, τ_X) and (Y, τ_Y) are homeomorphic, there is a homeomorphism $f : X \to Y$. But then f is continuous, bijective, and f^{-1} is continuous. But then f^{-1} is a homotopy inverse of f since it is continuous and $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$. But any continuous function is homotopic to itself, so if $f \circ f^{-1} = \mathrm{id}_Y$, then $f \circ f^{-1}$ is homotopic to id_Y. Similarly $f^{-1} \circ f$ is homotopic to id_X, and therefore f and f^{-1} are homotopy inverses of each other, meaning (X, τ_X) and (Y, τ_Y) are homotopy equivalent. □

This theorem does not reverse.

Theorem 3.7. *$\mathbb{R}^n$, with the standard topology, is homotopy equivalent to $\{ 0 \}$ with the subspace topology.*

Proof. Define $f : \mathbb{R}^n \to \{ 0 \}$ via $f(\mathbf{x}) = 0$. This is a constant function, so it is continuous. Let $g : \{ 0 \} \to \mathbb{R}^n$ be defined by $g(0) = \mathbf{0}$. Since g is a constant function, it is continuous. But $(g \circ f)(\mathbf{x}) = \mathbf{0}$, and this is homotopic to $\mathrm{id}_{\mathbb{R}^n}$ with the homotopy $H(\mathbf{x}, t) = t\mathbf{x}$. Also, $(f \circ g)(0) = 0$, so $f \circ g = \mathrm{id}_{\{ 0 \}}$, so $f \circ g$ is certainly homotopic to the identity since it is equal to it. Hence f and g are homotopy inverses of each other. □

This idea gets a name.

Definition 3.4 (**Contractible Topological Space**) A contractible topological space is a topological space (X, τ) that is homotopy equivalent to a single point $\{ 0 \}$. ■

$\mathbb{R}^n$ is not homeomorphic to a point, homeomorphisms must be bijective. This shows homotopy equivalent is much weaker. But even if (X, τ_X) and (Y, τ_Y) are topological spaces where X and Y have the same cardinality, it is possible for these spaces to be homotopy equivalent but not homeomorphic.

Let $X = \mathbb{R}^2 \setminus \{ \mathbf{0} \}$ be the punctured plane with the subspace topology. This has the same cardinality as $\mathbb{S}^1$ since both have the same cardinality as $\mathbb{R}$. They are not homeomorphic. The circle is compact by Heine-Borel, the punctured plane is not compact (also by Heine-Borel). Define $f : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{ \mathbf{0} \}$ to be the inclusion map, $f(\mathbf{x}) = \mathbf{x}$. Define $g : \mathbf{R}^2 \setminus \{ \mathbf{0} \} \to \mathbb{S}^1$ via $g(\mathbf{x}) = \mathbf{x}/||\mathbf{x}||_2$, the normalization map. Since $\mathbf{0} \notin \mathbb{R}^2 \setminus \{ \mathbf{0} \}$ this function is well-defined and continuous. We have $g \circ f$ is the identity function on $\mathbb{S}^1$, so it is homotopic to it. $f \circ g$ is the function sending $\mathbf{x} \neq \mathbf{0}$ to $\mathbf{x}/||\mathbf{x}||_2$. This is homotopic to the identity on $\mathbb{R}^2 \setminus \{ \mathbf{0} \}$, define H via:

$$H(\mathbf{x}, t) = (1-t)\frac{\mathbf{x}}{||\mathbf{x}||_2} + t\mathbf{x} \tag{37}$$

which is a homotopy between $f \circ g$ and the identity.

Let's modify our constraints. What if we have *compact* subsets of the plane? Could compact subsets of the same cardinality be homotopy equivalent but not homeomorphic? Consider $X \subseteq \mathbb{R}^2$ defined by:

$$X = \{ \mathbf{x} \in \mathbb{R}^2 \mid \frac{1}{2} \leq ||\mathbf{x}||_2 \leq 1 \} \tag{38}$$

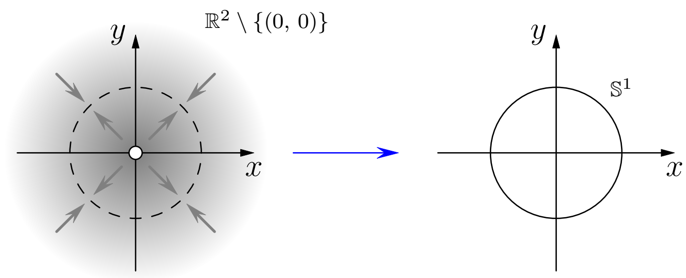


Figure 7: Homotopy Equivalence from the Punctured Plane to $\mathbb{S}^1$

This is the *closed annulus* in the plane. By Heine-Borel it is compact, and it too has the same cardinality as $\mathbb{S}^1$. The circle and the closed annulus are also homotopy equivalent, but not homeomorphic. To see this, intuively, if we remove two points from $\mathbb{S}^1$ we end up with two pieces. If we remove two pieces from X we still have one piece. The two spaces are homotopy equivalent, the same functions used with the punctured plane work. This is shown in Fig. 8.

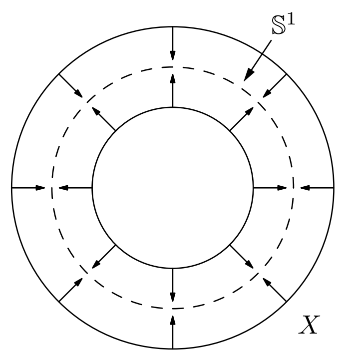


Figure 8: Homotopy Equivalence Between $\mathbb{S}^1$ and an Annulus

The annulus looks *two dimensional*, the circle is one dimensional (whatever this means). You modify your question. If both subsets are compact and have the same dimension, does homotopy equivalence imply homeomorphic? Great question! This is one of the most famous conjectures of topology, the Poincaré conjecture. If (X, τ) is a three dimensional manifold (locally the space looks just like $\mathbb{R}^3$) that is compact and homotopy equivalent to $\mathbb{S}^3$, the three dimensional sphere that lives as a subspace of $\mathbb{R}^4$, is (X, τ) homeomorphic to $\mathbb{S}^3$? The answer is yes, but this took about 100 years to solve.