Point-Set Topology: Lecture 18

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1 Regular and Normal Spaces

The Hausdorff condition is a very mild one and most spaces you'll encounter are Hausdorff. Metric spaces have far stronger separation properties, and these ideas are useful in the general topological setting as well. The new ideas are *regular*, *normal*, *completely Hausdorff*, *completely regular*, *completely normal*, and *perfectly normal*. In this section we'll discuss all of these ideas, show some relations between them, and draw some pictures.

Definition 1.1 (**Regular Topological Spaces**) A regular topological space is a topological space $(X,\,\tau)$ such that for all $x\in X$ and for all closed subsets $\mathcal{C}\subseteq X$ such that $x\notin\mathcal{C}$, there are open sets $\mathcal{U},\mathcal{V}\in\tau$ such that $x\in\mathcal{U}$, $\mathcal{C}\subseteq\mathcal{V}$, and $\mathcal{U}\cap\mathcal{V}=\emptyset$. ■

Note: in analysis authors usually define regular to mean *regular plus Fréchet*. That is, these authors state that a regular space is one where all singleton sets $\{\,x\,\}$ are closed, and such that all $x\in X$ and closed $\mathcal{C}\subseteq X$ can be separated by disjoint open sets. **We are not adopting this definition**. While many regular spaces that are studied happen to also be Hausdorff, there are also regular spaces that are not. If the space is Kolmogorov, however, then regular

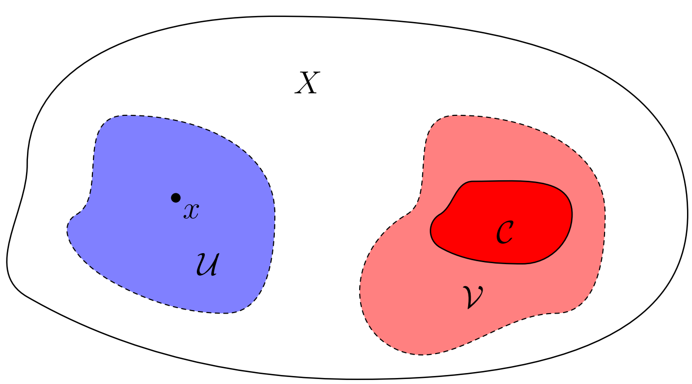

Figure 1: Regular Topological Space

implies Fréchet. From many weeks ago, a Kolmogorov space is a topological space (X, τ) such that for all $x, y \in X$ with $x \neq y$, there is an open set $\mathcal{U} \in \tau$ such that either $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, or $x \notin \mathcal{U}$ and $y \in \mathcal{U}$.

Theorem 1.1. *If (X, τ) is a topological space, then it is a Kolmogorov space if and only if for all $x, y \in X$ with $x \neq y$ there exists $\mathcal{C} \subseteq X$ that is closed such that either $x \in \mathcal{C}$ and $y \notin \mathcal{C}$, or $x \notin \mathcal{C}$ and $y \in \mathcal{C}$.*

Proof. Let $x, y \in X$ and $x \neq y$. Suppose (X, τ) is a Kolmogorov topological space. Then there is an open set $\mathcal{U} \in \tau$ such that either $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, or $x \notin \mathcal{U}$ and $y \in \mathcal{U}$. But then $\mathcal{C} = X \setminus \mathcal{U}$ is a closed set such that either $x \notin \mathcal{C}$ and $y \in \mathcal{C}$, or $x \in \mathcal{C}$ and $y \notin \mathcal{C}$. Going the other way, suppose (X, τ) is such that for all $x, y \in X$ with $x \neq y$ there is a closed set $\mathcal{C} \subseteq X$ such that either $x \in \mathcal{C}$ and $y \notin \mathcal{C}$, or $x \notin \mathcal{C}$ and $y \in \mathcal{C}$. Then $\mathcal{U} = X \setminus \mathcal{C}$ is an open set such that either $x \notin \mathcal{U}$ and $y \in \mathcal{U}$, or $x \in \mathcal{U}$ and $y \notin \mathcal{U}$. Hence, (X, τ) is a Kolmogorov topological space. □

Theorem 1.2. *If (X, τ) is a regular Kolmogorov topological space, then it is a Hausdorff topological space.*

Proof. Let $x, y \in X$, $x \neq y$. Since (X, τ) is Kolmogorov, there is a closed set $\mathcal{C} \subseteq X$ such that either $x \in \mathcal{C}$ and $y \notin \mathcal{C}$, or $x \notin \mathcal{C}$ and $y \in \mathcal{C}$. Suppose $x \notin \mathcal{C}$ and $y \in \mathcal{C}$, the proof is symmetric either way. But then $x \in X$ and $\mathcal{C} \subseteq X$ is a closed set such that $x \notin \mathcal{C}$. But (X, τ) is regular, so there are open sets $\mathcal{U}$, $\mathcal{V}$ such that $x \in \mathcal{U}$, $\mathcal{C} \subseteq \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. But $y \in \mathcal{C}$, so $y \in \mathcal{V}$. But then $\mathcal{U}$ and $\mathcal{V}$ are disjoint open sets separating x and y. Hence, (X, τ) is Hausdorff. □

Theorem 1.3. *If (X, τ) is a regular Kolmogorov topological space, then it is a Fréchet topological space.*

Proof. Since regular Kolmogorov spaces are Hausdorff, and Hausdorff spaces are Fréchet, we have that (X, τ) is a Fréchet topological space. □

Theorem 1.4. *If (X, τ) is a regular Fréchet topological space, then it is Hausdorff.*

Proof. Fréchet implies Kolmogorov, so (X, τ) is a regular Kolmogorov space, which is therefore Hausdorff. □

Example 1.1 The Kolmogorov property is a very mild one, which justifies some authors including it in the definition of regular. But, as we've defined, regular does not imply Hausdorff by itself. The set $\mathbb{Z}_2 = \{\,0,\,1\,\}$ with the indiscrete topology $\tau = \{\,\emptyset,\,\mathbb{Z}_2\,\}$ is not Hausdorff, not Fréchet, and not Kolmogorov, but it is regular. ■

Example 1.2 (**The Double Pointed Reals**) The double pointed real space is the topological space $\mathbb{R} \times \mathbb{Z}_2$, where $\mathbb{R}$ carries the standard Euclidean topology, and $\mathbb{Z}_2$ carries the indiscrete topology. Intuitively, for every real number $r \in \mathbb{R}$ you have another redundant copy r' that is topologically indistinguishable from r, even though r and r' are techinically different. This space is regular, but it is not Hausdorff. ■

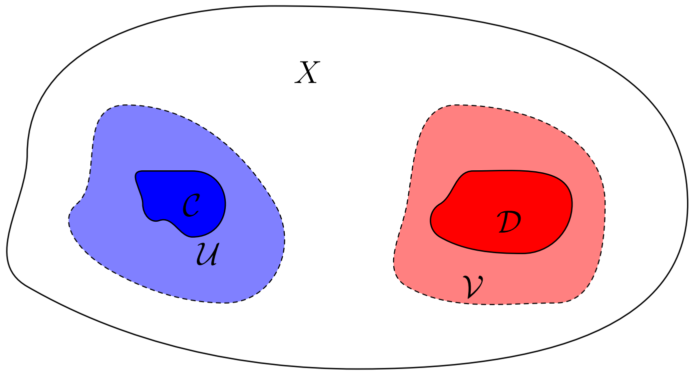

Figure 2: Normal Topological Space

Theorem 1.5. *If (X, τ) is a topological space, then it is regular if and only if for all $x \in X$ and $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$, there is an open set $\mathcal{V} \in \tau$ such that $x \in \mathcal{V}$ and $Cl_\tau(\mathcal{V}) \subseteq \mathcal{U}$.*

Proof. Suppose (X, τ) is regular and let $x \in X$ and $\mathcal{U} \in \tau$ be such that $x \in \mathcal{U}$. Since $\mathcal{U}$ is open, $X \setminus \mathcal{U}$ is closed. But $x \notin X \setminus \mathcal{U}$, so since (X, τ) is regular there exists $\mathcal{V}, \mathcal{W} \in \tau$ such that $x \in \mathcal{V}$, $X \setminus \mathcal{U} \subseteq \mathcal{W}$, and $\mathcal{V} \cap \mathcal{W} = \emptyset$. Now to prove that $\mathrm{Cl}_\tau(\mathcal{V}) \subseteq \mathcal{U}$. Since $\mathcal{W}$ is open and $X \setminus \mathcal{U} \subseteq \mathcal{W}$, we have that $X \setminus \mathcal{W}$ is closed and $X \setminus \mathcal{W} \subseteq \mathcal{U}$. But, since $\mathcal{V} \cap \mathcal{W} = \emptyset$, we have that $\mathcal{V} \subseteq X \setminus \mathcal{W}$. Hence $X \setminus \mathcal{W}$ is a closed set that contains $\mathcal{V}$ and sits inside of $\mathcal{U}$. But then $\mathrm{Cl}_\tau(\mathcal{V}) \subseteq X \setminus \mathcal{W}$, and hence $\mathrm{Cl}_\tau(\mathcal{V}) \subseteq \mathcal{U}$. Now, the other direction. Suppose for all $x \in X, \mathcal{U} \in \tau$ such that $x \in \mathcal{U}$, there is a $\mathcal{V} \in \tau$ with $x \in \mathcal{V}$ and $\mathrm{Cl}_\tau(\mathcal{V}) \subseteq \mathcal{U}$. Let $x \in X$ and $\mathcal{C} \subseteq X$ be closed and such that $x \notin \mathcal{C}$. Since $\mathcal{C}$ is closed, $X \setminus \mathcal{C}$ is open. But then there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $\mathrm{Cl}_\tau(\mathcal{U}) \subseteq X \setminus \mathcal{C}$. Let $\mathcal{V} = X \setminus \mathrm{Cl}_\tau(\mathcal{U})$. Then $\mathcal{V}$ is open since it is the complement of a closed set. But by definition $\mathcal{C} \subseteq \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Hence $\mathcal{U}$ and $\mathcal{V}$ are disjoint open sets that separate x and $\mathcal{C}$, so (X, τ) is regular. □

Definition 1.2 (**Normal Topological Space**) A normal topological space is a topological space (X, τ) such that for all disjoint closed subsets $\mathcal{C}, \mathcal{D} \subseteq X$, there are open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $\mathcal{C} \subseteq \mathcal{U}$, $\mathcal{D} \subseteq \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. ■

Theorem 1.6. *If (X, τ) is a second-countable and regular, then it is normal.*

Proof. Let $\mathcal{C}, \mathcal{D}$ be disjoint closed subsets of X. Since (X, τ) is second-countable there is a countable basis $\mathcal{B}$. For all $x \in \mathcal{C}$, since (X, τ) is regular, there is an open set $\mathcal{U}_x$ such that $x \in \mathcal{U}_x$ and $\mathcal{U}_x \cap \mathcal{D} = \emptyset$. But since (X, τ) is regular and $x \in \mathcal{U}_x$, there is a $\tilde{\mathcal{U}}_x$ such that $x \in \tilde{\mathcal{U}}_x$ and $\mathrm{Cl}_\tau(\tilde{\mathcal{U}}_x) \subseteq \mathcal{U}_x$. Similarly we can cover $\mathcal{D}$ with sets $\mathcal{V}_y$ and $\tilde{\mathcal{V}}_y$ such that for all $y \in \mathcal{D}$ we have $y \in \tilde{\mathcal{V}}_y$, $\mathrm{Cl}_\tau(\tilde{\mathcal{V}}_y) \subseteq \mathcal{V}$, and $\mathcal{V}_y \cap \mathcal{C} = \emptyset$. Let $\mathcal{O}_\mathcal{C}$ be defined by:

$$\mathcal{O}_\mathcal{C} = \{\, \mathcal{W} \in \mathcal{B} \mid \mathcal{W} \subseteq \tilde{\mathcal{U}}_x \text{ for some } x \in \mathcal{C} \,\} \tag{1}$$

and $\mathcal{O}_\mathcal{D}$ defined by:

$$\mathcal{O}_\mathcal{D} = \{\, \mathcal{W} \in \mathcal{B} \mid \mathcal{W} \subseteq \tilde{\mathcal{V}}_y \text{ for some } y \in \mathcal{D} \,\} \tag{2}$$

Since $\mathcal{B}$ is countable, both $\mathcal{O}_\mathcal{C}$ and $\mathcal{O}_\mathcal{D}$ are countable. But $\mathcal{B}$ is a basis, so $\mathcal{O}_\mathcal{C}$ and $\mathcal{O}_\mathcal{D}$ are open covers of $\mathcal{C}$ and $\mathcal{D}$, respectively. Let $\mathcal{U} : \mathbb{N} \to \mathcal{O}_\mathcal{C}$ and $\mathcal{V} : \mathbb{N} \to \mathcal{O}_\mathcal{D}$ be surjections. The union over all $\mathcal{U}_n$ covers $\mathcal{C}$, and the union over $\mathcal{V}_n$ covers $\mathcal{D}$, but it is possible for these unions to overlap. We make them disjoint as follows. Define $\mathcal{U}_n'$ by:

$$\mathcal{U}_n' = \mathcal{U}_n \setminus \bigcup_{k=0}^{n} \mathrm{Cl}_\tau(\mathcal{V}_k) \tag{3}$$

and $\mathcal{V}_n'$ via:

$$\mathcal{V}_n' = \mathcal{V}_n \setminus \bigcup_{k=0}^{n} \mathrm{Cl}_\tau(\mathcal{U}_k) \tag{4}$$

Then $\mathcal{U}_n'$ and $\mathcal{V}_n'$ are the difference of a closed set from an open set, and hence are all open. But now $\bigcup_n \mathcal{U}_n'$ and $\bigcup_n \mathcal{V}_n'$ are disjoint open sets that cover $\mathcal{C}$ and $\mathcal{D}$, respectively. Hence, $(X,\, \tau)$ is normal. □

In a just world, there would be three types of separation properties and two adjectives for these properties. There would be the three separation properties Hausdorff, regular, and normal. The three properties with the adjective *completely*. And the three properties with the adjective *perfectly*. The adjective *completely* should mean something similar for all three properties, and the adjective *perfectly* should mean something similar for all three properties as well. This is not the case, and life is not art, unfortunately.

In a just world, *completely* should mean every subspace has the property, *perfectly* should mean the separation property can replace open sets with continuous functions. There is a reason this is not done. If it were, Hausdorff and completely Hausdorff would mean the same thing, regular and completely regular would mean the same thing, and only normal and completely normal would be different ideas. Let's prove this. We've already done the Hausdorff case when we studied subspaces, but let's do it again. Why not.

Theorem 1.7. *If $(X,\, \tau)$ is a Hausdorff topological space, if $A \subseteq X$, and if τ_A is the subspace topology, then $(A,\, \tau_A)$ is a Hausdorff topological space.*

Proof. Let $x, y \in A$ with $x \neq y$. Then, since $A \subseteq X$, we have that $x, y \in X$ are distinct points. But $(X,\, \tau)$ is Hausdorff, so there are $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. But then $\tilde{\mathcal{U}} = A \cap \mathcal{U}$ and $\tilde{\mathcal{V}} = A \cap \mathcal{V}$ are open sets in τ_A by the definition of the subspace topology, $x \in \tilde{\mathcal{U}}$, $y \in \tilde{\mathcal{V}}$, and $\tilde{\mathcal{U}} \cap \tilde{\mathcal{V}} = \emptyset$. So $(A,\, \tau_A)$ is Hausdorff. □

Theorem 1.8. *If $(X,\, \tau)$ is a regular topological space, if $A \subseteq X$, and if τ_A is the subspace topology, then $(A,\, \tau_A)$ is regular.*

Proof. Let $x \in A$, $\mathcal{C} \subseteq A$ be closed with respect to τ_A, and $x \notin \mathcal{C}$. Then $A \setminus \mathcal{C}$ is open in τ_A, so there is a $\mathcal{U} \in \tau$ such that $A \setminus \mathcal{C} = A \cap \mathcal{U}$. Let $\tilde{\mathcal{C}} = \mathrm{Cl}_\tau(\mathcal{C})$. Note, this is closure with respect to τ, not τ_A. Since $x \in \mathcal{U}$ and $\mathcal{C} \cap \mathcal{U} = \emptyset$, we have that $x \notin \mathrm{Cl}_\tau(\mathcal{C})$, hence $x \notin \tilde{\mathcal{C}}$. But $(X,\,\tau)$ is regular and $\tilde{\mathcal{C}}$ is closed, being the closure of $\mathcal{C}$. So there are $\mathcal{V}, \mathcal{W} \in \tau$ such that $x \in \mathcal{V}$, $\tilde{\mathcal{C}} \subseteq \mathcal{W}$, and $\mathcal{V} \cap \mathcal{W} = \emptyset$. But then $\tilde{\mathcal{V}} = \mathcal{V} \cap A$ and $\tilde{\mathcal{W}} = \mathcal{W} \cap A$ are disjoint open sets in the subspace topology that separate x and $\mathcal{C}$. Hence, $(A,\,\tau_A)$ is regular. □

There is no identical theorem for normal spaces. A subspace of a normal space need not be normal. We give a new name to spaces with this property.

Definition 1.3 (**Completely Normal Topological Space**) A completely normal topological space is a topological space $(X,\,\tau)$ such that for all $A \subseteq X$ it is true that $(A,\,\tau_A)$ is normal, where τ_A is the subspace topology. ■

Theorem 1.9. *If $(X,\,\tau)$ is completely normal, then it is normal.*

Proof. If every subspace of $(X,\,\tau)$ is normal, then $(X,\,\tau)$ is normal since it is a subspace of itself. □

It is now extremely unfortunate that *completely* has a very different meaning when placed in front of the words Hausdorff and regular.

Definition 1.4 (**Completely Hausdorff Topological Space**) A completely Hausdorff topological space is a topological space $(X,\,\tau)$ such that for all $x, y \in X$ with $x \neq y$ there is a continuous function $f : X \to [0,\,1]$ where $[0,\,1]$ has the subspace topology, such that $f(x) = 0$ and $f(y) = 1$. That is, $x \in f^{-1}[\{\,0\,\}]$ and $y \in f^{-1}[\{\,1\,\}]$. ■

Theorem 1.10. *If $(X,\,\tau)$ is a completely Hausdorff topological space, then it is a Hausdorff topological space.*

Proof. Let $x, y \in X$, $x \neq y$, and let $f : X \to [0,\,1]$ be a continuous function such that $f(x) = 0$ and $f(y) = 1$. Let $\mathcal{U} = f^{-1}\big[[0,\,\frac{1}{4})\big]$ and $\mathcal{V} = f^{-1}\big[(\frac{3}{4},\,1]\big]$. Since $[0,\,\frac{1}{4})$ and $(\frac{3}{4},\,1]$ are open in the subspace topology, $\mathcal{U}$ and $\mathcal{V}$ are open. But also, by definition, $\mathcal{U} \cap \mathcal{V} = \emptyset$. Since $f(x) = 0$ we have $x \in \mathcal{U}$ and since $f(y) = 1$ we have $y \in \mathcal{V}$. So $(X,\,\tau)$ is Hausdorff. □

Definition 1.5 (**Completely Regular Topological Space**) A completely regular topological space is a topological space $(X,\,\tau)$ such that for all $x \in X$ and all closed $\mathcal{C} \subseteq X$ with $x \notin \mathcal{C}$ there is a continuous function $f : X \to [0,\,1]$, where $[0,\,1]$ has the subspace topology, such that $f(x) = 0$ and for all $y \in \mathcal{C}$ we have $f(y) = 1$. That is, $x \in f^{-1}[\{\,0\,\}]$ and $\mathcal{C} \subseteq f^{-1}[\{\,1\,\}]$. ■

Theorem 1.11. *If $(X,\,\tau)$ is a completely regular topological space, then it is regular.*

Proof. Let $x \in X$, $\mathcal{C} \subseteq X$ be closed, and $x \notin \mathcal{C}$. Since $(X,\,\tau)$ is completely regular there is a continuous function $f : X \to [0,\,1]$ such that $x \in f^{-1}[\{\,0\,\}]$ and $\mathcal{C} \subseteq f^{-1}[\{\,1\,\}]$. Let $\mathcal{U} = f^{-1}\big[[0,\,\frac{1}{4})\big]$ and $\mathcal{V} = f^{-1}\big[(\frac{3}{4},\,1]\big]$. Since $[0,\,\frac{1}{4})$ and

(X, τ)	**Hausdorff**	**Regular**	**Normal**
$-$	$x, y \in X$, $x \neq y$, there exists disjoint $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$ and $y \in \mathcal{V}$.	$x \in X$, $\mathcal{C} \subseteq X$ closed with $x \notin \mathcal{C}$, there exist disjoint $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{C} \subseteq \mathcal{V}$.	$\mathcal{C}, \mathcal{D} \subseteq X$ closed and disjoint, there exists disjoint $\mathcal{U}, \mathcal{V} \in \tau$ such that $\mathcal{C} \subseteq \mathcal{U}$ and $\mathcal{D} \subseteq \mathcal{V}$.
Completely	$x, y \in X$, $x \neq y$, there exists continuous $f : X \to [0,\,1]$ such that $f(x) = 0$ and $f(y) = 1$.	$x \in X$, $\mathcal{C} \subseteq X$ closed with $x \notin \mathcal{C}$, there exists continuous $f : X \to [0,\,1]$ with $f(x) = 0$ and $f[\mathcal{C}] = \{\,1\,\}$.	$A \subseteq X$, then (A, τ_A) is normal where τ_A is the subspace topology.
Perfectly	N/A	N/A	$\mathcal{C}, \mathcal{D} \subseteq X$ closed and disjoint, there exists continuous $f : X \to [0,\,1]$ such that $\mathcal{C} = f^{-1}[\{\,0\,\}]$ and $\mathcal{D} = f^{-1}[\{\,1\,\}]$.

Table 1: The Various Separation Properties

$(\frac{3}{4},\,1]$ are open in the subspace topology and f is continuous it is true that $\mathcal{U}$ and $\mathcal{V}$ are open. But by definition $\mathcal{U} \cap \mathcal{V} = \emptyset$. But also $x \in \mathcal{U}$ and $\mathcal{C} \subseteq \mathcal{V}$, so (X, τ) is regular. □

Now, isn't this quite dumb? *Completely* has one meaning for Hausdorff and regular, and an entirely different meaning for normal. Don't blame me, I didn't make the rules! This idea of separating things via continuous functions does have a name for normal spaces, but is slightly different.

Definition 1.6 (**Perfectly Normal Topological Space**) A perfectly normal topological space is a topological space (X, τ) such that for all disjoint closed sets $\mathcal{C}, \mathcal{D} \subseteq X$ there is a continuous function $f : X \to [0,\,1]$, where $[0,\,1]$ has the subspace topology, such that $f^{-1}[\{\,0\,\}] = \mathcal{C}$ and $f^{-1}[\{\,1\,\}] = \mathcal{D}$. ■

Perfectly normal means closed sets can be *precisely separated* by a continuous function. Contrast this with completely regular where it is only required that, given x and a closed set $\mathcal{C}$ with $x \notin \mathcal{C}$, that $x \in f^{-1}[\{\,0\,\}]$ and $\mathcal{C} \subseteq f^{-1}[\{\,1\,\}]$. With perfectly normal we require equality. There is no notion of this idea for regular and Hausdorff spaces (though *perfectly Hausdorff* and *perfectly regular* would be the likely candidate names). See Tab. 1 for an outline of the various ideas.