Point-Set Topology: Lecture 19

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1 Urysohn's Lemma and Metrization Theorem

We now come to one of the major theorems of point-set topology, the so-called *Urysohn Lemma*. The theorem deals with normal spaces and is used in the proof of one of the first metrization theorems.

Theorem 1.1 (**Urysohn's Lemma**). *If $(X,\,\tau)$ is a normal topological space, and if $\mathcal{C},\mathcal{D}\subseteq X$ are disjoint closed subsets, then there is a continuous function $f:X\rightarrow[0,\,1]$, where $[0,\,1]$ has the subspace topology, such that $f[\mathcal{C}]=\{\,0\,\}$ and $f[\mathcal{D}]=\{\,1\,\}$. That is, $\mathcal{C}\subseteq f^{-1}[\{\,0\,\}]$ and $\mathcal{D}\subseteq f^{-1}[\{\,1\,\}]$.*

Proof. Let $A=\mathbb{Q}\cap[0,\,1]$, the set of all rational numbers between 0 and 1, inclusive. Since $\mathbb{Q}$ is countable, A is countable as well. Moreover, A is countably infinite since it is not finite. Let $a:\mathbb{N}\rightarrow A$ be a bijection such that $a_0=0$ and $a_1=1$. We will now define open sets $\mathcal{U}_{a_n}$ such that whenever $a_m<a_n$ is true we have:

$$\mathrm{Cl}_\tau(\mathcal{U}_{a_m})\subseteq\mathcal{U}_{a_n}\tag{1}$$

To start, define:

$$\mathcal{U}_1=X\setminus\mathcal{D}\tag{2}$$

Since $(X,\,\tau)$ is normal and $\mathcal{C}\subseteq\mathcal{U}_1$ there is an open set $\mathcal{U}_0$ such that $\mathcal{C}\subseteq\mathcal{U}_0$ and $\mathrm{Cl}_\tau(\mathcal{U}_0)\subseteq\mathcal{U}_1$. Define $\mathcal{A}_N$ via:

$$\mathcal{A}_N=\{\,a_n\in A\mid n\in\mathbb{Z}_N\,\}\tag{3}$$

That is, the first N rational numbers given by the bijection $a:\mathbb{N}\rightarrow A$. We define $\mathcal{U}_{a_n}$ recursively. Suppose $\mathcal{U}_{a_n}\in\tau$ has been defined for all $n\in\mathbb{Z}_N$ such that $a_m<a_n$ implies $\mathrm{Cl}_\tau(\mathcal{U}_{a_m})\subseteq\mathcal{U}_{a_n}$. Since $\mathcal{A}_{N+1}$ is a subset of $\mathbb{Q}$, which is totally ordered, it is ordered as well. But it is also finite, and since $a_N\neq0$ and $a_N\neq1$, there are $a_m,a_n\in\mathcal{A}_{N+1}$ such that $a_m<a_N$ and $a_N<a_n$ where a_m is the largest such value and a_n is the smallest such value. But by the recursive definition $\mathrm{Cl}_\tau(\mathcal{U}_{a_m})\subseteq\mathcal{U}_{a_n}$. But $\mathrm{Cl}_\tau(\mathcal{U}_{a_m})$ is closed and $\mathcal{U}_{a_n}$ is open, so since $(X,\,\tau)$ is normal there is $\mathcal{U}_{a_N}\in\tau$ such that $\mathrm{Cl}_\tau(\mathcal{U}_{a_m})\subseteq\mathcal{U}_{a_N}$ and $\mathcal{U}_{a_N}\subseteq\mathcal{U}_{a_n}$. By the principle of induction, such a set exists for all a_n. That is, we now have for all rational numbers $p,q\in A$ with $p<q$, the following:

$$\mathrm{Cl}_\tau(\mathcal{U}_p)\subseteq\mathcal{U}_q\tag{4}$$

We extend this to all rationals as follows. Given $p \in \mathbb{Q}$, $p \notin A$, define:

$$\mathcal{U}_p = \begin{cases} X & p > 1 \\ \emptyset & p < 0 \end{cases} \tag{5}$$

Define $F : X \to \mathcal{P}(\mathbb{Q})$ via:

$$F(x) = \{\, p \in \mathbb{Q} \mid x \in \mathcal{U}_p \,\} \tag{6}$$

Define $f : X \to [0,\, 1]$ via:

$$f(x) = \inf\big(F(x)\big) \tag{7}$$

First, since A is bounded below by 0, $f(x)$ is well-defined for all $x \in X$. We now need to show that f is continuous, $f[\mathcal{C}] = \{\, 0 \,\}$, and $f[\mathcal{D}] = \{\, 1 \,\}$. First, $f[\mathcal{C}] = \{\, 0 \,\}$. If $x \in \mathcal{C}$, then $f(x) \in \mathcal{U}_0$ by definition of $\mathcal{U}_0$ (see above). Hence 0 is the smallest value $p \in \mathbb{Q}$ such that $f(x) \in \mathcal{U}_p$, and hence $f(x) = 0$. Next, $f[\mathcal{D}] = \{\, 1 \,\}$. By definition, for all $p \in \mathbb{Q}$ with $p \leq 1$, $f(x) \notin \mathcal{U}_p$. Hence $f(x) = \inf\big((1,\, \infty)\big) = 1$, so $f[\mathcal{D}] = \{\, 1 \,\}$. Lastly, we must prove f is continuous. This follows from the fact that A is a dense subset of $[0,\, 1]$. First, if $p \in \mathbb{Q}$ and $x \in \mathrm{Cl}_\tau(\mathcal{U}_p)$, then $f(x) \leq p$. This is true since for all $q \in \mathbb{Q}$ with $p < q$ we have $\mathcal{U}_p \subseteq \mathcal{U}_q$, and hence:

$$f(x) = \inf\big\{\, r \in \mathbb{Q} \mid f(x) \in \mathcal{U}_r \,\big\} \leq p \tag{8}$$

so $f(x) \leq p$. Next, if $x \notin \mathcal{U}_p$, then $f(x) \geq p$. Since $x \notin \mathcal{U}_p$, the only values $q \in \mathbb{Q}$ with $x \in \mathcal{U}_q$ must be greater than p, and hence $f(x) \geq p$. To conclude, a function is continuous if and only if for all $x \in X$ and all open $\mathcal{V}$ with $f(x) \in \mathcal{V}$ there is an open $\mathcal{U} \subseteq X$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$. Let $x \in X$ and $\mathcal{V} \subseteq \mathbb{R}$ be an open set such that $f(x) \in \mathcal{V}$. But $\mathcal{V}$ is open so there is an $\epsilon > 0$ such that $|y - f(x)| < \varepsilon$ implies $y \in \mathcal{V}$. Let $c = f(x) - \varepsilon/2$ and $d = f(x) + \varepsilon/2$. Let p and q be rational numbers such that $c < p < f(x) < q < d$. Define $\mathcal{U}$ via:

$$\mathcal{U} = \mathcal{U}_q \setminus \mathrm{Cl}_\tau(\mathcal{U}_p) \tag{9}$$

Then $\mathcal{U}$ is the difference of a closed set from an open set, and is hence open. By the above observation, for all $x_0 \in \mathcal{U}$ we have $p \leq f(x) \leq f(q_0)$, and hence $f[\mathcal{U}] \subseteq \mathcal{V}$. But also $x \in \mathcal{U}$ since $p < f(x) < q$. So f is continuous. $\square$

We get some use out of this immediately via Urysohn's metrization theorem.

Theorem 1.2 (**Urysohn's Metrization Theorem**). *If $(X,\, \tau)$ is a regular second countable Hausdorff topological space, then it is metrizable.*

Proof. Since $(X,\, \tau)$ is regular and second countable, it is normal. But also since $(X,\, \tau)$ is second countable there is a countable basis $\mathcal{B}$ for τ. Let $\mathcal{U} : \mathbb{N} \to \mathcal{B}$ be a surjection so that we may list the elements as:

$$\mathcal{B} = \{\, \mathcal{U}_0,\, \dots,\, \mathcal{U}_n,\, \dots \,\} \tag{10}$$

For all $m, n \in \mathbb{N}$ with $\mathrm{Cl}_\tau(\mathcal{U}_m) \subseteq \mathcal{U}_n$, by Urysohn's lemma there is a continuous function $g_{m,n} : X \to [0,\, 1]$ such that:

$$g_{m,n}[\mathrm{Cl}_\tau(\mathcal{U}_m)] = \{\, 1 \,\} \text{ and } g_{m,n}[X \setminus \mathcal{U}_n] = \{\, 0 \,\} \tag{11}$$

The set of all such $\mathcal{U}_m$ cover X since $\mathcal{B}$ is a basis and the set of all such functions is countable since the elements are indexed by $\mathbb{N} \times \mathbb{N}$. Relabel these functions as $f_n : X \to [0,\, 1]$ for all $n \in \mathbb{N}$. Define the function $F : X \to \mathbb{R}^\infty$ via:

$$F(x) = \big(f_0(x),\, \dots,\, f_n(x),\, \dots\big) \tag{12}$$

Since $\mathbb{R}^\infty$ has the product topology, and since each component function f_n is continuous, F is continuous. F is injective since given $x, y \in X$ with $x \neq y$ one can find a basis element $\mathcal{U}_n$ such that $x \in \mathcal{U}_n$ and $y \notin \mathcal{U}_n$, since $(X,\, \tau)$ is Hausdorff, but then there is a function f_n such that $f_n(x) = 1$ and $f_n(y) = 0$, hence $F(x) \neq F(y)$ since one of the components is different. Lastly, we must show F is a homeomorphism between $(X,\, \tau)$ and $(F[X],\, \tau_{\mathbb{R}^\infty_{F[X]}})$. Since $F : X \to \mathbb{R}^\infty$ is injective, $F : X \to F[X]$ is bijective. To show $F : X \to F[X]$ is a homeomorphism, since F is continuous, all that's left to show is that F is an open mapping. Let $\mathcal{U} \subseteq X$ be open, and given $y \in f[\mathcal{U}]$, let $x \in \mathcal{U}$ be such that $F(x) = y$. Since $x \in X$ there is an $n \in \mathbb{N}$ such that $f_n(x) > 0$ and $f_n[X \setminus \mathcal{U}] = \{\, 0 \,\}$. Let $\mathcal{V} = \mathrm{proj}_n^{-1}\big[(0,\, \infty)\big]$. Then, since projections are continuous and $(0,\, \infty)$ is open, $\mathcal{V} \subseteq \mathbb{R}^\infty$ is open. But then $f[X] \cap \mathcal{V}$ is open in $f[X]$ by definition of the subspace topology. But then $y \in f[X] \cap \mathcal{V}$, since:

$$\mathrm{proj}_n(y) = \mathrm{proj}_n\big(f(x)\big) = f_n(x) \tag{13}$$

and $f_n(x) > 0$, so $y \in \mathcal{V}$ by definition of $\mathcal{V}$. Lastly, $f[X] \cap \mathcal{V} \subseteq f[\mathcal{U}]$. For given $y \in f[X] \cap \mathcal{V}$, since $y \in f[X]$, there is some $x \in X$ such that $f(x) = y$. But if $y \in \mathcal{V}$, then $\mathrm{proj}_n(y) > 0$. But f_n is the zero function outside of $\mathcal{U}$, and hence $y \in f[\mathcal{U}]$. Since $f[\mathcal{U}]$ can be written as the union of all such $\mathcal{V}$, $f[\mathcal{U}]$ is open. That is, f is an open mapping with respect to the subspace topology on $f[X]$. Therefore $f : X \to \mathbb{R}^\infty$ is a topological embedding, meaning $(X,\, \tau)$ is homeomorphic to a subspace of a metrizable space, and is therefore metrizable. □

The last theorem to show is the Tietze extension theorem. It is logically equivalent to Urysohn's lemma.

Theorem 1.3 (**Tietze Extension Theorem**). *If $(X,\, \tau)$ is a normal topological space, if $\mathcal{C} \subseteq X$ is closed, and if $f : \mathcal{C} \to \mathbb{R}$ is continuous, then there is a continuous function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}|_A = f$, and if f is bounded, then $\tilde{f}$ is bounded as well with the same bounds.*

The proof is a bit lengthy, but I'd like to point out what this theorem does *not* say. It does not say $f : \mathcal{C} \to Y$ can be extended to all of X where $(Y,\, \tau_Y)$ is any topological space. This is false. One need look no further than the Euclidean plane. $\mathbb{S}^1$ is a closed subset of the Euclidean plane $\mathbb{R}^2$. The identity function

$f : \mathbb{S}^1 \to \mathbb{S}^1$ is continuous. However, there is **no** extension of this function to all of $\mathbb{R}^2$. To map $\mathbb{R}^2$ to $\mathbb{S}^1$ while keeping $\mathbb{S}^1$ fixed means, intuitively, we'd need to tear the plane at some point. This is not continuous. Imagine you had a lump of dough in the shape of a disk. How would you push the inside of the lump of dough to the outside to make a circle? You'd need to press your fingers through the dough and make a hole. The Tietze extension theorem is only applicable when the co-domain is $\mathbb{R}$.