Point-Set Topology: Lecture 21

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1 Connected and Path Connected Components

A topological space $(X,\,\tau)$ does not need to be connected, but it can always be divided into connected parts. These parts are called the *connected components* of the space.

Definition 1.1 (**Connected Component**) The connected component of a point $x\in X$ in a topological space $(X,\,\tau)$ is the set $C\subseteq X$ defined by:

$$C=\bigcup\{\,A\subseteq X\mid x\in A\text{ and }A\text{ is connected}\,\}\tag{1}$$

That is, the *largest* connected subset of X containing x. ■

Theorem 1.1. *If $(X,\,\tau)$ is a topological space, if $x\in X$, and if $C\subseteq X$ is the connected component of x, then $(C,\,\tau_C)$ is a connected topological space where τ_C is the subspace topology.*

Proof. For if not then there are disjoint non-empty open subsets $\mathcal{U},\mathcal{V}$ such that $C=\mathcal{U}\cup\mathcal{V}$. But since $x\in C$ either $x\in\mathcal{U}$ or $x\in\mathcal{V}$. Suppose $x\in\mathcal{U}$. Let $\mathcal{O}$ be defined by:

$$\mathcal{O}=\{\,A\subseteq X\mid x\in A\text{ and }A\text{ is connected}\,\}\tag{2}$$

Then by definition of connected components $C=\bigcup\mathcal{O}$. Suppose there is some $A\in\mathcal{O}$ such that $\mathcal{V}\cap A\neq\emptyset$. But then $A\cap\mathcal{V}$ and $A\cap\mathcal{U}$ are non-empty open subsets of A with respect to the subspace topology τ_A. But $A\cap\mathcal{V}$ and $A\cap\mathcal{U}$ are disjoint since $\mathcal{U}$ and $\mathcal{V}$ are, and hence $(A,\,\tau_A)$ can be separated by open sets, which is a contradiction since $A\in\mathcal{O}$ and hence $(A,\,\tau_A)$ is connected. We have thus shown that $\mathcal{V}\cap A=\emptyset$ for every $A\in\mathcal{O}$, and hence $\mathcal{V}\cap\bigcup\mathcal{O}=\emptyset$. But $\mathcal{V}\subseteq C$ and $C=\bigcup\mathcal{O}$, meaning $\mathcal{V}=\emptyset$, which is a contradiction since $\mathcal{V}$ is non-empty. So $(C,\,\tau_C)$ is connected. □

There is no analogous theorem for intersections. The intersection of two connected sets does not need to be connected. See Fig. 1. Even if we are given infinitely many connected sets $\mathcal{U}_n$ that are all nested, $\mathcal{U}_{n+1}\subseteq\mathcal{U}_n$, the intersection $\bigcap_{n\in\mathbb{N}}\mathcal{U}_n$ need not be connected. Take $\mathcal{U}_n$ to be all points $(x,\,y)$ such that $y=0$, $y=1$, or $0<y<1$ and $x\geq n$. Each $\mathcal{U}_n$ is connected, and $\mathcal{U}_{n+1}\subseteq\mathcal{U}_n$,

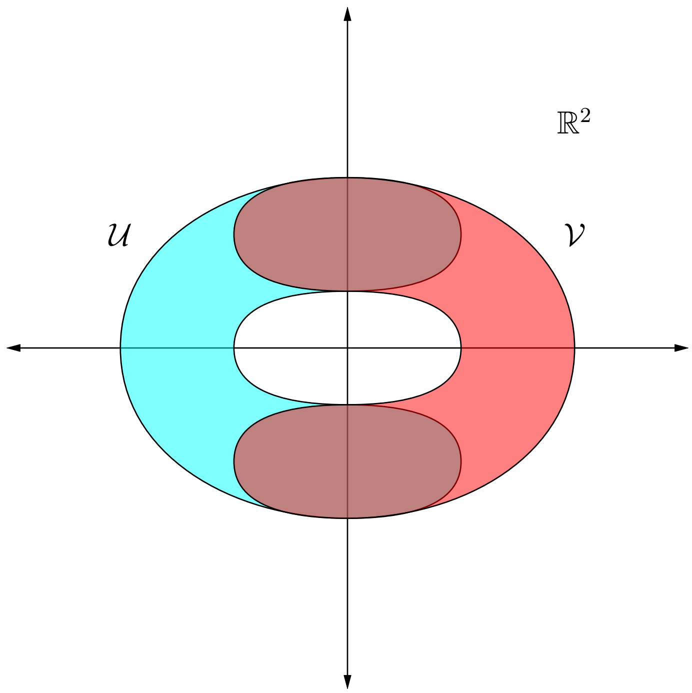

Figure 1: The Intersection of Connected Sets

but $\bigcap_n \mathcal{U}_n$ is two separated lines, the lines $y = 0$ and $y = 1$, which is not connected.

As another example, take $\mathcal{C}_n$ to be the closed box $[-1, 1] \times [-n, n]$ and let $\mathcal{U}_n = \mathbb{R}^2 \setminus \mathcal{C}_n$. Then all of the sets $\mathcal{U}_n$ are open, and are nested and connected, however $\bigcap \mathcal{U}_n$ splits the plane in half.

Connected components are closed. To show this we'll need to following theorem about subspaces.

Theorem 1.2. *If (X, τ) is a topological space, if $\mathcal{C} \subseteq X$ is closed, and if $A \subseteq \mathcal{C}$ is closed with respect to the subspace topology $\tau_{\mathcal{C}}$, then A is closed with respect to τ.*

Proof. Since $A \subseteq \mathcal{C}$ is closed with respect to $\tau_{\mathcal{C}}$, $\mathcal{C} \setminus A$ is open. But if $\mathcal{C} \setminus A$ is open, then by the definition of the subspace topology there is an open set $\mathcal{U} \in \tau$ such that $\mathcal{C} \setminus A = \mathcal{C} \cap \mathcal{U}$. But then:

$$A = \mathcal{C} \setminus (\mathcal{C} \cap \mathcal{U}) = (\mathcal{C} \setminus \mathcal{C}) \cup (\mathcal{C} \setminus \mathcal{U}) = \emptyset \cup (\mathcal{C} \setminus \mathcal{U}) = \mathcal{C} \setminus \mathcal{U} \tag{3}$$

But $\mathcal{C} \subseteq X$ is closed, so $\mathcal{C} \setminus \mathcal{U}$ is the difference of an open set from a closed set which is therefore closed. Therefore A is closed with respect to τ. $\square$

Theorem 1.3. *If (X, τ) is a topological space, if $A \subseteq X$, if τ_A is the subspace topology, and if (A, τ_A) is connected, then $(Cl_\tau(A), \tau_{Cl_\tau(A)})$ is connected.*

Proof. Suppose not. Then there are disjoint closed subsets $\mathcal{C}, \mathcal{D} \subseteq \mathrm{Cl}_\tau(A)$ such that $\mathcal{C} \cup \mathcal{D} = \mathrm{Cl}_\tau(C)$. But $\mathrm{Cl}_\tau(A)$ is closed with respect to τ, so $\mathcal{C}$ and $\mathcal{D}$ are also closed with respect to τ. But then $\mathcal{U} = X \setminus \mathcal{C}$ and $\mathcal{V} = X \setminus \mathcal{D}$ are disjoint open sets in τ. But then $\mathcal{U} \cap A$ and $\mathcal{V} \cap A$ are disjoint non-empty open subsets with respect to τ_A that separate A, a contradiction since (A, τ_A) is connected. Hence, $(\mathrm{Cl}_\tau(A), \tau_{\mathrm{Cl}_\tau(A)})$ is connected. $\square$

Theorem 1.4. *If (X, τ) is a topological space, and if $C \subseteq X$ is the connected component of $x \in X$, then it is closed.*

Proof. We have that $C \subseteq \mathrm{Cl}_\tau(C)$ by the definition of closure. But since connected components are connected we have that $\mathrm{Cl}_\tau(C)$ is connected. But $x \in \mathrm{Cl}_\tau(C)$ so, by the definition of connected components, $\mathrm{Cl}_\tau(C) \subseteq C$. Hence $C = \mathrm{Cl}_\tau(C)$ meaning C is closed. $\square$

The idea of connected components allows us to define totally disconnected spaces.

Definition 1.2 (**Totally Disconnected Topological Space**) A totally disconnected topological space is a topological space (X, τ) such that for all $x \in X$ the connected component of x is the set $C = \{ x \}$. That is, the connected components of the space are singleton sets. $\blacksquare$

Theorem 1.5. *If $\tau_{\mathbb{Q}}$ is the subspace topology of $\mathbb{Q}$ with respect to the standard Euclidean topology $\tau_{\mathbb{R}}$, then $(\mathbb{Q}, \tau_{\mathbb{Q}})$ is totally disconnected.*

Proof. For let $x \in \mathbb{Q}$ and let C be the connected component of x. Suppose $y \in \mathbb{Q}$ is such that $y \in C$ with $y \neq x$. Suppose $x < y$ (the idea is symmetric either way). Since $x, y \in \mathbb{Q}$, and since $\mathbb{Q} \subseteq \mathbb{R}$, it is true that $x, y \in \mathbb{R}$. Let $z \in \mathbb{R} \setminus \mathbb{Q}$ be such that $x < z$ and $z < y$. This is possible since x and y are real numbers and the irrational numbers are dense in $\mathbb{R}$. Let $\mathcal{U} = \mathbb{Q} \cap (-\infty, z)$ and $\mathcal{V} = \mathbb{Q} \cap (z, \infty)$. Then $\mathcal{U}$ and $\mathcal{V}$ are open in the subspace topology. But moreover $\mathcal{U}$ and $\mathcal{V}$ separate x and y, meaning $y \notin C$. Hence C is just a singleton set. $\square$

Theorem 1.6. *If (X, τ) is a totally disconnected topological space, then it is a Fréchet topological space.*

Proof. For all $x \in X$, since (X, τ) is totally disconnected, the connected component of x is $C = \{ x \}$. But connected components are closed, hence $\{ x \}$ is closed. But this is true for all $x \in X$, hence (X, τ) is a Frëchet topological space. $\square$

Example 1.1 (**The Rational Bug-Eyed Line**) Let $X = \mathbb{Q} \times \{ 0, 1 \}$, and give this the product topology τ where $\mathbb{Q}$ has the subspace topology from $\mathbb{R}$ and $\mathbb{Z}_2$ has the discrete topology. Define the equivalence relation R on X to be the equivalence relation induced by $(x,\, 0)R(x,\, 1)$ for all $x \in \mathbb{Q}$ except for $x = 0$. The quotient space $(X/R,\, \tau_{X/R})$ is like the bug-eyed line, but has only rational points (and the two origins). This space is totally disconnected, but not Hausdorff. ■

Path connected components can be similarly defined.

Definition 1.3 (**Path Connected Component**) The path connected component of a point $x \in X$ in a topological space $(X,\, \tau)$ is the set $C \subseteq X$ defined by:

$$C = \big\{\, y \in X \mid \text{There is a path from } x \text{ to } y \,\big\} \tag{4}$$

That is, the set of all points that can be connected by a path to x. ■

Theorem 1.7. *If $(X,\, \tau)$ is a topological space, if $x \in X$, and if C is the path-connected component of x, then $(C,\, \tau_C)$ is connected.*

Proof. By definition of path connected component, $(C,\, \tau_C)$ forms a path connected space, and path connected spaces are connected. □

2 Locally Connected and Locally Path Connected

Some warning. In topology the word *locally* has two possible meanings. Given some *property*, locally could mean:

1. For every point x in the space there is some open or closed set A containing the point such that A has the desired property.
2. There exists a basis of open sets $\mathcal{B}$ such that every element of $\mathcal{B}$ has the desired property.

For example, a *locally metrizable* space $(X,\, \tau)$ is a topological space such that for all $x \in X$ there is an open set $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U},\, \tau_{\mathcal{U}})$ is a metrizable topological space.

Locally compact means that for all $x \in X$ there is an open set $\mathcal{U}$ and a compact subspace $(K,\, \tau_X)$ such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq K$.

With the first use of the word *locally*, if a space has the property *globally*, then it automatically has it locally. That is, metrizable spaces are locally metrizable, and compact spaces are locally compact. With the second use of the word we are not so lucky. For connectedness, *locally* connected and *locally* path connected mean we can describe the topology using connected or path connected open subsets.

The weird connected examples that are not path connected fail to be *locally connected* and *locally path connected*. These two properties have some very useful results associated with them, so we take the time to study these ideas.

Definition 2.1 (**Locally Connected Topological Space**) A locally connected topological space is a topological space (X, τ) such that there is a basis $\mathcal{B}$ for τ such that for all $\mathcal{U} \in \mathcal{B}$ the subspace $(\mathcal{U}, \tau_{\mathcal{U}})$ is connected. ■

Example 2.1 Neither the infinite broom nor the topologist's sine curve are locally connected, though both are connected. For the topologist's sine curve, any sufficiently small (small makes sense, this is a metric space since its a subset of $\mathbb{R}^2$) open set containing $(0, 0)$ must contain disconnected pieces of the graph of $f(x) = \sin(1/x)$ as well, meaning there can be no basis of connected subsets. A similar argument holds for the infinite broom. ■

In any space connected components are closed. In locally connected spaces they are also open.

Theorem 2.1. *If (X, τ) is a locally connected topological space, if $x \in X$, and if C is the connected component of x, then $C \in \tau$.*

Proof. Since (X, τ) is locally connected there is a basis of connected subsets $\mathcal{B}$. Given $y \in C$, since $\mathcal{B}$ is a basis it is an open cover, so there is a $\mathcal{U}_y \in \mathcal{B}$ such that $y \in \mathcal{U}_y$. But then C and $\mathcal{U}_y$ are connected subsets that both contain y, so $C \cup \mathcal{U}_y$ is connected. But C is the connected component of x, so since $C \cup \mathcal{U}_y$ is connected and $x \in C \cup \mathcal{U}_y$ it must be true that $C \cup \mathcal{U}_y \subseteq C$. But $C \subseteq C \cup \mathcal{U}_y$ by definition of unions, so $C = C \cup \mathcal{U}_y$ and therefore $\mathcal{U}_y \subseteq C$. But C can be written as the union of all such $\mathcal{U}_y$ for all $y \in C$, meaning C is the union of open sets, which is therefore open. □

Theorem 2.2. *If (X, τ) is a totally disconnected and locally connected topological space, then $\tau = \mathcal{P}(X)$.*

Proof. Since (X, τ) is totally disconnected, connected components are singleton sets. But since (X, τ) is locally connected, connected components are open. Hence for all $x \in X$ the set $\{\, x \,\}$ is open, meaning all subsets of X are open. That is, $\tau = \mathcal{P}(X)$. □

There is a path connected analogue to locally connected spaces.

Definition 2.2 (**Locally Path Connected Topological Space**) A locally path connected topological space is a topological space (X, τ) such that there is a basis $\mathcal{B}$ for τ such that for all $\mathcal{U}$ the subspace $(\mathcal{U}, \tau_{\mathcal{U}})$ is path connected. ■

Theorem 2.3. *If (X, τ) is locally path connected, if $x \in X$, and if $C \subseteq X$ is the path connected component of x, then C is open.*

Proof. Since (X, τ) is locally path connected there is a basis $\mathcal{B}$ of open path connected subspaces. But since $\mathcal{B}$ is a basis it is an open cover, so given $y \in C$ there is a $\mathcal{U}_y \in \mathcal{B}$ such that $y \in \mathcal{U}_y$. But C and $\mathcal{U}_y$ are path connected and $y \in \mathcal{U}_y \cap C$, so $\mathcal{U}_y \cup C$ is path connected. Since C is the path connected component of x, $\mathcal{U}_y \cup C \subseteq C$, and hence $\mathcal{U}_y \subseteq C$. But then C can be written as the union of all such $\mathcal{U}_y$ for all $y \in C$, meaning C is the union of open sets, and is therefore open. □

Theorem 2.4. *If (X, τ) is locally path connected, if $x \in X$, and if $C \subseteq X$ is the path connected component of x, then C is closed.*

Proof. Suppose not. Then $C \neq \mathrm{Cl}_\tau(C)$. Let $y \in \mathrm{Cl}_\tau(C) \setminus C$. Since (X, τ) is locally path connected there is a basis $\mathcal{B}$ of open path connected subspaces. But if $\mathcal{B}$ is a basis, since $y \in X$ there is a $\mathcal{U} \in \mathcal{B}$ such that $y \in \mathcal{U}$. But if $y \in \mathrm{Cl}_\tau(X)$ and $y \in \mathcal{U}$, then $\mathcal{U} \cap C$ is non-empty (from the definition of closure). But $\mathcal{U}$ and C are path connected subspaces with non-empty intersection, so $\mathcal{U} \cup C$ is path connected. But C is the path connected component of x, so $\mathcal{U} \cup C \subseteq C$, and hence $\mathcal{U} \subseteq C$. But then $y \in C$, a contradiction. Hence, C is closed. $\square$

Theorem 2.5. *If (X, τ) is connected and locally path connected, then it is path connected.*

Proof. Suppose not. Then there are $x, y \in X$ such that there is no continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Let C be the path connected component of x. Then $y \notin C$, and hence C is a proper subset of X. But $x \in C$, so C is non-empty. But since C is the path connected component of x, and since (X, τ) is locally path connected, C is both open and closed. But then C is a non-empty proper subset of X that is both open and closed, and therefore (X, τ) is disconnected, which is a contradiction since (X, τ) is connected. Hence, (X, τ) is path connected. $\square$

3 Arc Connected

Arc connected is a slightly stronger notion that many might think is the same thing as path connected. In Hausdorff spaces, the notions are the same. First, a definition.

Definition 3.1 (**Arc Connected Topological Space**) An arc connected topological space is a topological space (X, τ) such that for all $x, y \in X$ there is an injective continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. $\blacksquare$

The only difference between path connected and arc connected is the introduction of the word *injective*. The following theorem takes a lot of effort, and we don't have time for it, but I still want to present it. The proof is omitted.

Theorem 3.1. *If (X, τ) is a Hausdorff path connected topological space, then it is arc connected.*

Example 3.1 Without the Hausdorff condition you can have path connected spaces that are not arc connected. The bug-eyed line is an example. There is a path from the first zero 0_0 to the second zero 0_1, namely:

$$f(x) = \begin{cases} 0_0 & t = 0 \\ t - t^2 & 0 < t < 1 \\ 0_1 & t = 1 \end{cases} \tag{5}$$

That is, you start at the first zero, walk out a bit to some positive real number, and then walk back to the second zero. This is not injective, you crossed over a bunch of real numbers twice. It's impossible to do this injectively. This space is not Hausdorff, so there is no violation of the previous theorem. ■