Point-Set Topology: Lecture 22

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1 Compactness

For a course in point-set topology, if you understand the general notions (topological spaces, continuity, Hausdorffness, sequentialness), the basis properties (first and second countable), creating new spaces (products, subspaces, quotients), the separation ideas (regular and normal), connectedness, and compactness, then you have an absolutely solid understanding of general topology. We've covered all of these ideas except compactness, which we've only discussed in the context of metric spaces (or *metrizable* spaces). We now take the time to study compactness in the general topological setting.

In a metric space we proved several theorems about compactness, primarily the Bolzano-Weierstrass, Heine-Borel, generalized Heine-Borel, and equivalence of compactness theorems. This told us that compactness can be described by sequences and by open sets. In the topological setting it is more natural to define compactness via open sets.

Definition 1.1 (**Compact Topological Space**) A compact topological space is a topological space $(X,\,\tau)$ such that for all open covers $\mathcal{O}\subseteq\tau$ there is a finite subset $\Delta\subseteq\mathcal{O}$ such that Δ is open cover. ■

We have spent a lot of time on compactness in the setting of metric spaces. Let's not waste that time, and copy over some of the theorems but rephrase them for *metrizable* spaces.

Theorem 1.1. *If $(X,\,\tau)$ is a metrizable topological space, then it is compact if and only if for all metrics d on X that induce τ, $(X,\,d)$ is a compact metric space.*

Proof. By the equivalence of compactness theorem, any metric d that induces τ has the property that any open cover of open sets in the metric space $(X,\,d)$ has a finite open subcover, which is precisely the definition of compactness in the topological setting. □

Theorem 1.2. *If $(X,\,\tau)$ is a metrizable topological space, then it is compact if and only if for every metric d on X that induces τ, $(X,\,d)$ is complete and totally bounded.*

Proof. This follows from the previous theorem and the generalized Heine-Borel theorem. $\square$

Theorem 1.3. *If* $(\mathbb{R},\,\tau_{\mathbb{R}})$ *is the standard Euclidean line, and if* $A\subseteq\mathbb{R}$*, then* $(A,\,\tau_{\mathbb{R}_A})$ *is compact if and only if* A *is closed and bounded.*

Proof. The Euclidean topology on $\mathbb{R}$, $\tau_{\mathbb{R}}$, is induced by the Euclidean metric $d(x,\,y)=|x-y|$. The result then follows from the Heine-Borel theorem. $\square$

Theorem 1.4. *If* $(\mathbb{R}^n,\,\tau_{\mathbb{R}^n})$ *is Euclidean space, and if* $A\subseteq\mathbb{R}^n$*, then* $(A,\,\tau_{\mathbb{R}^n_A})$ *is compact if and only if* A *is closed and bounded.*

Proof. This too follows from the Heine-Borel theorem. $\square$

This now gives us plenty of familiar spaces that are compact. Lacking a metrizable space, there are still plenty of pleasing properties compact topologies yield.

Theorem 1.5. *If* $(X,\,\tau)$ *is a compact topological space, and if* $\mathcal{C}\subseteq X$ *is closed, then* $(\mathcal{C},\,\tau_{\mathcal{C}})$ *is compact where* $\tau_{\mathcal{C}}$ *is the subspace topology.*

Proof. For if not then there is an open cover $\mathcal{O}_{\mathcal{C}}$ of $(\mathcal{C},\,\tau_{\mathcal{C}})$ with no finite subcover. But by the definition of the subspace topology, for all $\mathcal{U}\in\mathcal{O}_{\mathcal{C}}$ there is an open set $\tilde{\mathcal{U}}\in\tau$ such that $\mathcal{U}=\mathcal{C}\cap\tilde{\mathcal{U}}$. Let $\mathcal{O}_X\subseteq\tau$ be defined by:

$$\mathcal{O}_X=\{\,\tilde{\mathcal{U}}\in\tau\mid\mathcal{U}\in\mathcal{O}_{\mathcal{C}}\,\}\tag{1}$$

$\mathcal{O}_X$ covers $\mathcal{C}$ with elements of τ, but it need not cover all of X. However, since $\mathcal{C}$ is closed, $X\setminus\mathcal{C}$ is open. Let $\mathcal{O}\subseteq\tau$ be defined by:

$$\mathcal{O}=\mathcal{O}_X\cup\{\,X\setminus\mathcal{C}\,\}\tag{2}$$

Then $\mathcal{O}\subseteq\tau$ is an open cover of X, and since $(X,\,\tau)$ is compact there is a finite subcover Δ. Define $\Delta_X=\Delta\setminus\{\,X\setminus\mathcal{C}\,\}$. Then, by definition of $\mathcal{O}$ and $\mathcal{O}_X$, $\Delta_X\subseteq\mathcal{O}_X$. But also Δ_X is finite. But since Δ covers X and $X\setminus\mathcal{C}$ is disjoint from $\mathcal{C}$, Δ_X must cover $\mathcal{C}$ as well. Define $\Delta_{\mathcal{C}}$ via:

$$\Delta_{\mathcal{C}}=\{\,\mathcal{U}\cap\mathcal{C}\mid\mathcal{U}\in\Delta_X\,\}\tag{3}$$

By definition of $\mathcal{O}_X$ and Δ_X we have that $\Delta_{\mathcal{C}}\subseteq\mathcal{O}_{\mathcal{C}}$. But since Δ_X covers $\mathcal{C}$, so does $\Delta_{\mathcal{C}}$. But then $\Delta_{\mathcal{C}}\subseteq\mathcal{O}_{\mathcal{C}}$ is a finite subset that still covers $\mathcal{C}$, a contradiction. Hence, $(\mathcal{C},\,\tau_{\mathcal{C}})$ is compact. $\square$

This theorem does not need to reverse, in general. That is, compact subsets don't need to be closed (but in metric spaces they are). Give $\mathbb{R}$ the indiscrete topology $\tau=\{\,\emptyset,\,\mathbb{R}\,\}$. Then every subset $A\subseteq\mathbb{R}$ is compact since the only open covers possible are finite (they have at most two subsets). However only $\emptyset$ and $\mathbb{R}$ are closed. If we add the Hausdorff condition, then compact subspaces are closed.

Before proving this, it was quite annoying dealing with the subspace topology in the previous theorem. It feels unnecessary. With compact subspaces, if we can cover the subspace with sets that are open in the *ambient space*, then there is a finite subcover of this as well. Let's prove this.

Theorem 1.6. *If $(X,\,\tau)$ is a topological space, if $A\subseteq X$, and if $(A,\,\tau_A)$ is compact, where τ_A is the subspace topology, then for all $\mathcal{O}\subseteq\tau$ such that $A\subseteq\bigcup\mathcal{O}$, there is a finite subset $\Delta\subseteq\mathcal{O}$ such that $A\subseteq\bigcup\Delta$.*

Proof. If not, then there is a subset $\mathcal{O}\subseteq\tau$ such that $A\subseteq\bigcup\mathcal{O}$ but with no finite subset that still covers A. Let $\tilde{\mathcal{O}}$ be defined by:

$$\tilde{\mathcal{O}}=\{\,\mathcal{U}\cap A\mid A\in\mathcal{O}\,\}\tag{4}$$

By definition of the subspace topology, $\tilde{\mathcal{O}}\subseteq\tau_A$. But since $A\subseteq\bigcup\mathcal{O}$, we have $A=\bigcup\tilde{\mathcal{O}}$. But $(A,\,\tau_A)$ is compact, so there is a finite subcover $\tilde{\Delta}\subseteq\tilde{\mathcal{O}}$. Since it is finite we may label it:

$$\tilde{\Delta}=\{\,\mathcal{U}_0\cap A,\,\dots,\,\mathcal{U}_n\cap A\,\}\tag{5}$$

Define $\Delta\subseteq\mathcal{O}$ via:

$$\Delta=\{\,\mathcal{U}_0,\,\dots,\,\mathcal{U}_n\,\}\tag{6}$$

Then we have:

$$A=\bigcup_{k=0}^{n}(\mathcal{U}_k\cap A)=\Big(\bigcup_{k=0}^{n}\mathcal{U}_n\Big)\cap A\subseteq\bigcup_{k=0}^{n}\mathcal{U}_n\tag{7}$$

So $\Delta\subseteq\mathcal{O}$ is a finite subset of $\mathcal{O}$ that covers A, which is a contradiction. Hence, for any $\mathcal{O}\subseteq\tau$ such that $A\subseteq\bigcup\mathcal{O}$, there is a finite subset $\Delta\subseteq\mathcal{O}$ such that $A\subseteq\bigcup\Delta$. □

Now when dealing with compact subspaces we can restrict our attention to open sets in the ambient space, which is often easier.

Theorem 1.7. *If $(X,\,\tau)$ is a Hausdorff topological space, if $A\subseteq X$, and if $(A,\,\tau_A)$ is compact, where τ_A is the subspace topology, then A is closed.*

Proof. If $A=\emptyset$, then there is nothing to prove since $\emptyset$ is closed. Suppose $A\ne\emptyset$. If A is not closed, then $X\setminus A$ is not open, and hence there is an $x\in X\setminus A$ such that for all $\mathcal{U}\in\tau$ with $x\in\mathcal{U}$ it is not true that $\mathcal{U}\subseteq X\setminus A$ (otherwise we could write $X\setminus A$ as the union of all such $\mathcal{U}$ for all $x\in X\setminus A$, showing that $X\setminus A$ is the union of open sets, which is therefore open). But then for all $y\in A$, since $(X,\,\tau)$ is Hausdorff, there exist open sets $\mathcal{U}_y,\mathcal{V}_y$ such that $x\in\mathcal{U}_y$, $y\in\mathcal{V}_y$, and $\mathcal{U}_y\cap\mathcal{V}_y=\emptyset$. But the collection of all such $\mathcal{V}_y$ cover A, and since $(A,\,\tau_A)$ is compact, there is a finite subcover. Label the elements of the finite subcover as $\mathcal{V}_0,\,\dots,\,\mathcal{V}_n$. Label the corresponding open sets around x as $\mathcal{U}_0,\,\dots,\,\mathcal{U}_n$. Define $\tilde{\mathcal{U}}$ via:

$$\tilde{\mathcal{U}}=\bigcap_{k=0}^{n}\mathcal{U}_k\tag{8}$$

Then $\tilde{\mathcal{U}}$ is open, being the intersection of finitely many open sets, and $x\in\tilde{\mathcal{U}}$ since $x\in\mathcal{U}_k$ for all k. But $\tilde{\mathcal{U}}$ is disjoint from A. For if $y\in A$ and $y\in\tilde{\mathcal{U}}$, since $\mathcal{V}_0,\,\dots,\,\mathcal{V}_n$ cover A there is some $0\le k\le n$ such that $y\in\mathcal{V}_k$. But then:

$$\tilde{\mathcal{U}}\cap\mathcal{V}_k\subseteq\mathcal{U}_k\cap\mathcal{V}_k=\emptyset\tag{9}$$

a contradiction, so $\tilde{\mathcal{U}}$ and A are disjoint. But then $\tilde{\mathcal{U}}$ is an open set such that $x \in \tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}} \subseteq X \setminus A$, which is a contradiction. Hence, A is closed. □

Theorem 1.8. *If $(X,\,\tau_X)$ is a compact topological space, if $(Y,\,\tau_Y)$ is a topological space, and if $f : X \to Y$ is continuous, then $(f[X],\,\tau_{Y_{f[X]}})$ is compact where $\tau_{Y_{f[X]}}$ is the subspace topology.*

Proof. Suppose not and let $\mathcal{O}$ be an open cover of $f[X]$ with no finite subcover. Then for all $\mathcal{V} \in \mathcal{O}$, by the definition of the subspace topology, there is an open $\tilde{\mathcal{V}} \in \tau_Y$ such that $\mathcal{V} = \tilde{\mathcal{V}} \cap f[X]$. Define $\tilde{\mathcal{O}}$ via:

$$\tilde{\mathcal{O}} = \{\, \tilde{\mathcal{V}} \mid \mathcal{V} \in \mathcal{O} \,\} \tag{10}$$

Then $\tilde{\mathcal{O}}$ is a collection of open sets in Y that cover $f[X]$. Since f is continuous, for all $\tilde{\mathcal{V}} \in \tilde{\mathcal{O}}$ the set $f^{-1}[\tilde{\mathcal{V}}]$ is open in X. But $\tilde{\mathcal{O}}$ covers $f[X]$, and hence the set:

$$\mathscr{O} = \{\, f^{-1}[\tilde{\mathcal{V}}] \mid \tilde{\mathcal{V}} \in \tilde{\mathcal{O}} \,\} \tag{11}$$

is an open cover of $(X,\,\tau_X)$. But $(X,\,\tau_X)$ is compact, so there is a finite subcover $\mathscr{D} \subseteq \mathscr{O}$. Form the set $\tilde{\Delta} \subseteq \tilde{\mathcal{O}}$ by choosing a single element $\tilde{\mathcal{V}} \in \tilde{\mathcal{O}}$ for each $\mathcal{U} \in \mathscr{D}$ such that $\mathcal{U} = f^{-1}[\tilde{\mathcal{V}}]$. Then $\tilde{\Delta} \subseteq \tilde{\mathcal{O}}$ is a finite subset that covers $f[X]$. But then the set Δ of sets of the form $\tilde{\mathcal{V}} \cap f[X]$ for all $\tilde{\mathcal{V}} \in \tilde{\Delta}$ is a finite subset of $\mathcal{O}$ that covers $f[X]$, a contradiction. Hence, $(f[X],\,\tau_{Y_{f[X]}})$ is compact. □

Theorem 1.9. *if $(X,\,\tau_X)$ is a compact topological space, if $(Y,\,\tau_Y)$ is a Hausdorff topological space, and if $f : X \to Y$ is continuous and bijective, then f is a homeomorphism.*

Proof. It suffices to show that f is a closed mapping since f is a homeomorphism if and only if it is bijective, continuous, and a closed mapping. Since f is bijective and continuous by hypothesis, we need only show it is also a closed mapping. Let $\mathcal{C} \subseteq X$ be closed. But $(X,\,\tau_X)$ is compact and $\mathcal{C}$ is closed, so $(\mathcal{C},\,\tau_{X_{\mathcal{C}}})$ is compact. But then, since f is continuous, $f[\mathcal{C}] \subseteq Y$ is a compact subspace. But $(Y,\,\tau_Y)$ is Hausdorff, so $f[\mathcal{C}]$ is closed. Hence, f is a closed mapping, so it is a homeomorphism. □

Theorem 1.10. *If $(X,\,\tau)$ is a compact Hausdorff space, then it is regular.*

Proof. Let $x \in X$, $\mathcal{C} \subseteq X$ be closed, and $x \notin \mathcal{C}$. We must find open subsets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $\mathcal{C} \subseteq \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Since $x \notin \mathcal{C}$, for all $y \in \mathcal{C}$ we have $x \neq y$. But $(X,\,\tau)$ is Hausdorff so for all $y \in \mathcal{C}$ there are open sets $\mathcal{U}_y$, $\mathcal{V}_y$ such that $x \in \mathcal{U}_y$, $y \in \mathcal{V}_y$, and $\mathcal{U}_y \cap \mathcal{V}_y = \emptyset$. But then the set:

$$\mathcal{O} = \{\, \mathcal{V}_y \mid y \in \mathcal{C} \,\} \tag{12}$$

is a collection of open sets that cover $\mathcal{C}$. But $(X,\,\tau)$ is compact, and $\mathcal{C}$ is a closed subset, meaning there is a finite subset $\Delta \subseteq \mathcal{O}$ that covers $\mathcal{C}$. Label the sets as:

$$\Delta = \{\, \mathcal{V}_0,\, \dots,\, \mathcal{V}_N \,\} \tag{13}$$

Label the corresponding open sets around x similarly:

$$\Lambda = \{\,\mathcal{U}_0,\,\ldots,\,\mathcal{U}_N\,\} \tag{14}$$

Define:

$$\mathcal{W} = \bigcap \Lambda = \bigcap_{k=0}^{N} \mathcal{U}_k \tag{15}$$

Then $\mathcal{W}$ is open, being the intersection of finitely many open sets, and $x \in \mathcal{W}$ since $x \in \mathcal{U}_k$ for each k. Furthermore, define:

$$\mathcal{E} = \bigcup \Delta = \bigcup_{k=0}^{N} \mathcal{V}_k \tag{16}$$

Since Δ covers $\mathcal{C}$, we have $\mathcal{C} \subseteq \mathcal{E}$. Morever, $\mathcal{E}$ is open, being the union of open sets. So $x \in \mathcal{W}$ and $\mathcal{C} \subseteq \mathcal{E}$, and both $\mathcal{W}$ and $\mathcal{E}$ are open. We conclude by showing that $\mathcal{W} \cap \mathcal{E} = \emptyset$. By definition of $\mathcal{W}$ we have that $\mathcal{W} \subseteq \mathcal{U}_k$ for each k. But $\mathcal{U}_k \cap \mathcal{V}_k = \emptyset$. So $\mathcal{W} \cap \mathcal{V}_k = \emptyset$ for all k, and hence $\mathcal{W} \cap \mathcal{E} = \emptyset$. So $(X,\,\tau)$ is regular. □

Theorem 1.11. *If $(X,\,\tau)$ is a compact Hausdorff space, then it is normal.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be closed disjoint subsets of X. If one of them is empty, we may choose $\mathcal{U} = \emptyset$ and $\mathcal{V} = X$. So suppose neither are empty. By the previous theorem, a compact Hausdorff space is regular. Hence for all $x \in \mathcal{C}$, since $x \notin \mathcal{D}$, we have that there are open disjoint sets $\mathcal{U}_x, \mathcal{V}_x \in \tau$ such that $x \in \mathcal{U}_x$, $\mathcal{C} \subseteq \mathcal{V}_x$, and $\mathcal{U}_x \cap \mathcal{V}_x = \emptyset$. The collection:

$$\mathcal{O} = \{\,\mathcal{U}_x \mid x \in \mathcal{C}\,\} \tag{17}$$

is a collection of open sets that cover $\mathcal{C}$. Since $(X,\,\tau)$ is compact and $\mathcal{C} \subseteq X$ is closed there is a finite subcover $\Delta \subseteq \mathcal{O}$ of $\mathcal{C}$. Label the elements as:

$$\Delta = \{\,\mathcal{U}_0,\,\ldots,\,\mathcal{U}_N\,\} \tag{18}$$

label the corresponding open sets around $\mathcal{D}$ as well:

$$\Lambda = \{\,\mathcal{V}_0,\,\ldots,\,\mathcal{V}_N\,\} \tag{19}$$

Define:

$$\mathcal{W} = \bigcap \Lambda = \bigcap_{k=0}^{N} \mathcal{V}_k \tag{20}$$

Then $\mathcal{W}$ is open, being the intersection of finitely many open sets, and $\mathcal{D} \subseteq \mathcal{W}$ since $\mathcal{D} \subseteq \mathcal{V}_k$ for each k. Furthermore, define:

$$\mathcal{E} = \bigcup \Delta = \bigcup_{k=0}^{N} \mathcal{U}_k \tag{21}$$

Then $\mathcal{E}$ is open, being the union of open sets, and $\mathcal{C} \subseteq \mathcal{E}$ since Δ is an open cover of $\mathcal{C}$. Finally, $\mathcal{W} \cap \mathcal{E} = \emptyset$ since $\mathcal{U}_k \cap \mathcal{V}_k = \emptyset$ for all k, and hence $\mathcal{W} \cap \mathcal{V}_k = \emptyset$ as well, meaning $\mathcal{W} \cap \mathcal{E} = \emptyset$. Therefore $(X,\,\tau)$ is normal. □

Definition 1.2 (**Sequentially Compact Topological Space**) A sequentially compact topological space is a topological space $(X,\,\tau)$ such that for every sequence $a:\mathbb{N}\rightarrow X$ there is a convergent subsequence a_k. ■

Theorem 1.12. *If $(X,\,\tau)$ is metrizable, then it is compact if and only if it is sequentially compact.*

Proof. This follows from the equivalence of compactness theorem. □

Metrizable, *sequentially compact*, and *compact* are three properties such that none implies the other. This is a good counterexample to the bad practice many students often make in logic. If P, Q, and R are statements, and if $P\land Q\Leftrightarrow P\land Q$, is it true that $Q\Leftrightarrow R$? That is, can you *divide* by P? These three topological properties provide a counterexample. Metrizable and compact if and only if metrizable and sequentially compact. Let's show none of these statements, by themselves, are logically equivalent or imply any of the others.

- Compact and not metrizable: The indiscrete topology on $\mathbb{R}$. The only open covers possible are finite to begin with, so the space is compact. It is not metrizable since it is not Hausdorff.
- Sequentially compact and not metrizable: The Sierpinski space $(\mathbb{Z}_2,\,\tau)$ where $\tau=\big\{\,\emptyset,\,\{\,0\,\},\,\mathbb{Z}_2\,\big\}$. Any sequence $a:\mathbb{N}\rightarrow\mathbb{Z}_2$ must have a convergent subsequence since either infinitely many indices $n\in\mathbb{N}$ are such that $a_n=0$ or infinitely many are such that $a_n=1$ (since $\mathbb{N}$ is infinite). Hence there must be a constant subsequence, which is a convergent one. The space is not metrizable since it is not Hausdorff.
- Metrizable and not compact: The discrete topology on $\mathbb{R}$. The cover consisting of all single points $\{\,x\,\}$ for $x\in\mathbb{R}$ has no finite subcover. It doesn't even have a countable subcover.
- Metrizable and not sequentially compact: The standard topology on $\mathbb{R}$. The sequence $a:\mathbb{N}\rightarrow\mathbb{R}$ defined by $a_n=n$ has no convergent subsequence.
- Compact and not sequentially compact: The product space $\prod_{r\in[0,\,1]}[0,\,1]$. This is compact, with the product topology, by the Tychonoff theorem, something we'll get to soon. It is not sequentially compact. The product is uncountable so the space is not first countable, and intuitively sequences are not enough to describe the space.
- Sequentially compact and not compact: The long line. It is not compact, take as your open cover open intervals about the center that get larger and larger and exhaust the space. The open sets in this cover are nested, and no finite collection of such intervals cover the space. It is sequentially compact, which is hard to imagine. For simplicity, let's just use the long ray, which is the product of the first uncountable ordinal ω^1 with $[0,\,1)$ equipped with the lexicographic order topology. Given a sequence a in the long ray, $a_n=(\alpha_n,\,x_n)$ where α_n is an element of the first uncountable

ordinal and $x_n \in [0, 1)$. But the first uncountable ordinal is uncountable and $\mathbb{N}$ is countable, so this sequence cannot exhaust all of ω^1. Because of this the elements will be contained in a small subset of the long ray, a subset that *looks* like the closed unit interval [0, 1], topologically speaking. By using an argument similar to the proof of Bolzano's theorem (which is used to prove the Heine-Borel theorem for $\mathbb{R}$), we can conclude that any such sequence must have a convergent subsequence. So the long ray is sequentially compact.