Point-Set Topology: Lecture 23

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1 Other Ideas for Compactness

Compact and sequentially compact are two of the most desirable properties used in analysis, geometry, and topology. There are several weaker notions that have found there way into several branches of mathematics. These properties are weaker, but are satisfied by many more spaces. *Paracompactness*, for example, is a particularly weak property that has enormous use in manifold theory and geometry, and every metric space is paracompact (even though your average metric space is not compact).

Definition 1.1 (**Limit Point Compact Topogical Space**) A limit point compact topological space is a topological space $(X,\,\tau)$ such that for all infinite subsets $A\subseteq X$ there exists a point $x\in X$ such that for all $\mathcal{U}\in\tau$ with $x\in\mathcal{U}$, there is a $y\in A$ such that $y\neq x$ and $y\in\mathcal{U}$ ■

Limit point compact was the original defining property of compactness when mathematicians were first thinking about the topology of the real line. Unlike compactness and sequential compactness, where one can't really say one idea is *stronger* than the other, limit point compactness is a weaker notion. For the real line, however, limit point compact is equivalent to compactness (this is a corollary of the Bolzano-Weierstrass theorem).

Theorem 1.1. *If $(X,\,\tau)$ is a sequentially compact topological space, then it is limit point compact.*

Proof. For if not, then there is an infinite set $A\subseteq X$ such that for all $x\in X$ there is a $\mathcal{U}\in\tau$ such that $x\in\mathcal{U}$ and for $y\in A$ either $y=x$ or $y\notin\mathcal{U}$. But if A is infinite, there is a countably infinite subset $B\subseteq A$. Let $a:\mathbb{N}\to B$ be a bijection. But $(X,\,\tau)$ is sequentially compact, so there is a convergent subsequence a_k. Let $x\in X$ be the limit. Then there is a $\mathcal{U}\in\tau$ such that $x\in\mathcal{U}$ and for all $y\in A$ either $y=x$ or $y\notin\mathcal{U}$. But $a_{k_n}\to x$ and $x\in\mathcal{U}$, so there is an $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ with $n>N$ we have $a_{k_n}\in\mathcal{U}$. But $a:\mathbb{N}\to B$ is bijective and $k:\mathbb{N}\to\mathbb{N}$ is strictly increasing, so a_k is injective, meaning for all $n>N$, a_{k_n} are distinct elements of B, and hence A, that are contained in $\mathcal{U}$, which is a contradiction. So $(X,\,\tau)$ is limit point compact. □

Compact also implies limit point compact, but a weaker notion than compact also implies limit point compact. This weaker notion is occasionally useful.

Definition 1.2 (**Countably Compact Topological Space**) A countably compact topological space is a topological space $(X,\,\tau)$ such that for all countable open covers $\mathcal{O}\subseteq\tau$ there is a finite subset $\Delta\subseteq\mathcal{O}$ such that Δ is an open cover. ■

Theorem 1.2. *If $(X,\,\tau)$ is compact, then it is countably compact.*

Proof. Any countable open cover is indeed an open cover, and since $(X,\,\tau)$ is compact, there must be a finite subcover. □

Theorem 1.3. *If $(X,\,\tau)$ is countably compact, then it is limit point compact.*

Proof. If not there is an infinite subset $A\subseteq X$ such that for all $x\in X$ there is a $\mathcal{U}\in\tau$ such that for all $y\in A$ either $y=x$ or $y\notin\mathcal{U}$. But since A is infinite, there is a countably infinite subset $B\subseteq A$. Let $a:\mathbb{N}\rightarrow B$ be a bijection. But then for all $x\in X$ there is a $\mathcal{U}\in\tau$ such that for all $y\in B$ either $y=x$ or $y\notin\mathcal{U}$. But then $\mathrm{Cl}_\tau(B)=B$. For suppose not and let $y\in\mathrm{Cl}_\tau(B)\setminus B$. Then there is an open subset $\mathcal{U}\in\tau$ such that $y\in\mathcal{U}$ and for all $z\in B$ either $z=y$ or $z\notin\mathcal{U}$. But $y\notin B$, so $z\neq y$, and hence $\mathcal{U}\cap B=\emptyset$. But then, since $\mathcal{U}$ is open, $\mathrm{Cl}_\tau(B)\subseteq X\setminus\mathcal{U}$ and $y\notin X\setminus\mathcal{U}$, a contradiction. So $\mathrm{Cl}_\tau(B)=B$ and B is closed. For all $n\in\mathbb{N}$ let $\mathcal{U}_{n+1}\in\tau$ be such that $\mathcal{U}_{n+1}\cap B=\{\,a_n\,\}$. Let $\mathcal{U}_0=X\setminus\{\,B\,\}$. Then:

$$\mathcal{O}=\{\,\mathcal{U}_n\mid n\in\mathbb{N}\,\}\tag{1}$$

is a countable open cover of $(X,\,\tau)$. But since $(X,\,\tau)$ is countably compact, there is a finite subcover Δ. But then there is a $\mathcal{U}_n$ such that infinitely many elements of B are contained inside $\mathcal{U}_n$, which is a contradiction. So $(X,\,\tau)$ is limit point compact. □

Theorem 1.4. *If $(X,\,\tau)$ is compact, then it is limit point compact.*

Proof. If $(X,\,\tau)$ is compact, then it is countably compact, and hence $(X,\,\tau)$ is limit point compact. □

Theorem 1.5. *If $(X,\,\tau)$ if a limit point compact Frëchet topological space, then it is countably compact.*

Proof. Suppose not, and let $\mathcal{O}\subseteq X$ be a countably infinite open cover that has no finite subcover. Since $\mathcal{O}$ is countably infinite there is a bijection $\mathcal{U}:\mathbb{N}\rightarrow\mathcal{O}$. Define $\mathcal{V}_n$ to be:

$$\mathcal{V}_n=\bigcup_{k=0}^{n}\mathcal{U}_k\tag{2}$$

Since $\mathcal{O}$ has no finite subcover, $\mathcal{V}_n\neq X$ for all $n\in\mathbb{N}$. But then $X\setminus\mathcal{V}_n$ must be infinite for each $n\in\mathbb{N}$, so we can pick an injective sequence $a:\mathbb{N}\rightarrow X$ such that $a_n \notin \mathcal{V}_n$ for all $n \in \mathbb{N}$. That is, $k \leq n$ implies $a_n \notin \mathcal{U}_k$. Let A be defined by:

$$A = \{\, a_n \in X \mid n \in \mathbb{N} \,\} \tag{3}$$

Then A is infinite, and since $(X,\, \tau)$ is limit point compact, there is a point $x \in X$ such that for all $\mathcal{W} \in \tau$ with $x \in \mathcal{W}$ there is a $y \in A$ such that $y \neq x$ and $y \in \mathcal{W}$. That is, there is some $n \in \mathbb{N}$ such that $a_n \neq x$ and $a_n \in \mathcal{W}$. Since $\mathcal{O}$ is an open cover there is a set $\mathcal{U}_N \in \mathcal{O}$ such that $x \in \mathcal{U}_N$. But since $(X,\, \tau)$ is Fréchet, the singleton sets are closed. But then the set:

$$\mathcal{C} = \{\, a_k \in X \mid a_k \neq x \text{ and } k \leq N \,\} \tag{4}$$

is the union of finitely many points, and is hence closed. But $x \notin \mathcal{C}$ by definition, and hence $x \in X \setminus \mathcal{C}$. Moreover, since $\mathcal{C}$ is closed, $X \setminus \mathcal{C}$ is open. But then $\mathcal{U}_N \cap (X \setminus \mathcal{C})$ is an open set containing x. But then there is an $n \in \mathbb{N}$ such that $a_n \neq x$ and $a_n \in \mathcal{U}_N \cap (X \setminus \mathcal{C})$. From the definition of $\mathcal{C}$ it must be true that $n > N$. But then $a_n \in \mathcal{U}_N$ and $n > N$, a contradiction since $N \leq n$ implies $a_n \notin \mathcal{U}_N$. So $(X,\, \tau)$ is countably compact. $\square$

Sequentially compact is a nice property, and it is quite a shame compactness does not imply it in general. It is also a shame sequential compactness does not imply compact. If we add *sequential* to our hypothesis, we can get one direction to work. First, a little lemma.

Theorem 1.6. *If $(X,\, \tau)$ is countably compact, and if $\mathcal{C} \subseteq X$ is closed, then $(\mathcal{C},\, \tau_{\mathcal{C}})$ is countably compact.*

Proof. The proof is a mimicry of the idea for compact spaces. Given a countable open cover of $\mathcal{C}$, by adding $X \setminus \mathcal{C}$ we obtain a countable open cover of X since $\mathcal{C}$ is closed, so $X \setminus \mathcal{C}$ is open, and adding one more set to a countable collection is still countable. But since $(X,\, \tau)$ is countably compact there is a finite subcover. Restricting this finite subcover to $\mathcal{C}$ shows that $(\mathcal{C},\, \tau_{\mathcal{C}})$ is countably compact. $\square$

Theorem 1.7. *If $(X,\, \tau)$ is a sequential countably compact topological space, then it is sequentially compact.*

Proof. For if not, then there is a sequence $a : \mathbb{N} \to X$ with no convergent subsequence. Let $A \subseteq X$ be defined by:

$$A = \bigcup_{n \in \mathbb{N}} \mathrm{Cl}_\tau\big(\{\, a_n \,\}\big) \tag{5}$$

Then A is sequentially closed. For if $b : \mathbb{N} \to A$ is a sequence that converges to $y \in X$, either there is an $m \in \mathbb{N}$ such that $y \in \mathrm{Cl}_\tau(\{\, a_m \,\})$ or not. If there is such an m, then $y \in A$. If not, then by choosing a_k and b_ℓ to be subsequences such that $b_{\ell_n} \in \mathrm{Cl}_\tau(\{\, a_{k_n} \,\})$, we have found a convergent subsequence $a_{k_n} \to y$, which is a contradiction. Hence A is sequentially closed. But $(X,\, \tau)$ is sequential, so A is closed. But then $(A,\, \tau_A)$ is countably compact, where τ_A is the subspace topology. But countably compact implies limit point compact, so there is a point $x \in A$ such that for all $\mathcal{U} \in \tau_A$ with $x \in \mathcal{U}$, there is a a_n such that $a_n \neq x$ and $a_n \in \mathcal{U}$. But x must be in only finitely many sets of the form $\mathrm{Cl}_\tau\big(\{\, a_n \,\}\big)$, otherwise a would have a convergent subsequence converging to x. But then there is an $N \in \mathbb{N}$ such that for all $n > N$ we have $x \notin \mathrm{Cl}_\tau\big(\{\, a_n \,\}\big)$. But then, defining:

$$B = \bigcup_{n=N+1}^{\infty} \mathrm{Cl}_\tau\big(\{\, a_n \,\}\big) \tag{6}$$

we see that B is closed, by the previous argument, but B does not contain the point x, which is a contradiction since x is still a limit point of B. So $(X,\,\tau)$ is sequentially compact. $\square$

A short corollary of this is often used when sequential compactness is desired.

Theorem 1.8. *If $(X,\,\tau)$ is compact and first countable, then it is sequentially compact.*

Proof. Compact implies countably compact, and first countable implies sequential. So $(X,\,\tau)$ is countably compact and sequential, and therefore sequentially compact. $\square$

While sequentially compact does not imply compact, there is a partial result. Sequentially compact always implies countably compact, and often enough countably compact is sufficient.

Theorem 1.9. *If $(X,\,\tau)$ is sequentially compact, then it is countably compact.*

Proof. If not there is a countably infinite open cover $\mathcal{O} \subseteq \tau$ with no finite subcover. But then, since $\mathcal{O}$ is countably infinite, there is a bijection $\mathcal{U} : \mathbb{N} \to \mathcal{O}$ so that we may list the elements as:

$$\mathcal{O} = \{\, \mathcal{U}_0,\, \dots,\, \mathcal{U}_n,\, \dots \,\} \tag{7}$$

But $\mathcal{U}_n \neq X$ for all $n \in \mathbb{N}$, otherwise $\Delta = \{\, \mathcal{U}_n \,\}$ is a finite subcover. So $X \setminus \mathcal{U}_n \neq \emptyset$ for all $n \in \mathbb{N}$. Moreover, the set $\mathcal{V}_n$ defined by:

$$\mathcal{V}_n = \bigcup_{k=0}^{n} \mathcal{U}_n \tag{8}$$

is such that $\mathcal{V}_n \neq X$, otherwise $\mathcal{O}$ has a finite subcover. Define $a : \mathbb{N} \to X$ via $a_n \in X \setminus \mathcal{V}_n$ for all $n \in \mathbb{N}$. But $(X,\,\tau)$ is sequentially compact, so there is a convergent subsequence a_k with limit $x \in X$. But since $\mathcal{O}$ covers X, there is a $\mathcal{U}_N$ such that $x \in \mathcal{U}_N$. But then for all $n > N$ we have $a_{k_n} \notin \mathcal{U}_N$, which is a contradiction since $a_{k_n} \to x$. So $(X,\,\tau)$ is countably compact. $\square$

One way to weaken compactness is by lessening open covers to countable open covers. The other way is by lessening finite subcover to countable subcover. This idea has proven quite useful in many applications in analysis.

Definition 1.3 (**Lindelöf Topological Space**) A Lindelöf topological space is a topological space $(X,\,\tau)$ such that for every open cover $\mathcal{O} \subseteq \tau$ there is a countable subcover $\Delta \subseteq \mathcal{O}$. ■

Theorem 1.10. *If $(X,\,\tau)$ is a topological space, then it is countably compact and Lindelöf if and only if it is compact.*

Proof. Compact implies countably compact, and it also implies Lindelöf since every open has a finite open subcover, which is definitely a countable open subcover. Going the other, if we are given $\mathcal{O} \subseteq \tau$ an open cover, since $(X,\,\tau)$ is Lindelöf there is a countable subcover $\tilde{\Delta} \subseteq \mathcal{O}$. But $(X,\,\tau)$ is countably compact, so if $\tilde{\Delta}$ is a countable open cover, then there is a finite subcover $\Delta \subseteq \tilde{\Delta}$. But then $\Delta \subseteq \mathcal{O}$ is a finite subcover, so $(X,\,\tau)$ is compact. □

Theorem 1.11. *If $(X,\,\tau)$ is second countable, then it is Lindelöf.*

Proof. If not, there is an open cover $\mathcal{O} \subseteq \tau$ with no countable subcover. Since $(X,\,\tau)$ is second countable there is a countable basis $\mathcal{B}$. Let $\mathcal{U} : \mathbb{N} \to \mathcal{B}$ be a surjection:

$$\mathcal{B} = \{\,\mathcal{U}_0,\,\dots,\,\mathcal{U}_n,\,\dots\} \tag{9}$$

Define $A \subseteq \mathbb{N}$ via:

$$A = \{\,n \in \mathbb{N} \mid \text{there exists } \mathcal{V} \in \mathcal{O} \text{ such that } \mathcal{U}_n \subseteq \mathcal{V}\,\} \tag{10}$$

A is non-empty since $\mathcal{B}$ is a basis, and hence for all $\mathcal{V} \in \mathcal{O}$ there is some $\mathcal{U}_n \in \mathcal{B}$ such that $\mathcal{U}_n \subseteq \mathcal{V}$. Since $\mathcal{O}$ is not countable, it is certainly not finite, and hence not empty, showing that A is non-empty as well. Since $A \subseteq \mathbb{N}$ it is countable as well. By the axiom of choice we can find a function $\mathcal{V} : A \to \mathcal{O}$ such that for all $n \in A$, $\mathcal{U}_n \subseteq \mathcal{V}_n$. But then the set $\Delta \subseteq \mathcal{O}$ defined by:

$$\Delta = \{\,\mathcal{V}_n \mid n \in \mathbb{N}\,\} \tag{11}$$

is a countable open cover of $(X,\,\tau)$. For given $x \in X$, since $\mathcal{O}$ is an open cover there is a $\mathcal{W} \in \mathcal{O}$ such that $x \in \mathcal{W}$. But $\mathcal{B}$ is a basis, so there is a $\mathcal{U}_n \in \mathcal{B}$ such that $x \in \mathcal{U}_n$ and $\mathcal{U}_n \subseteq \mathcal{W}$. But then $\mathcal{U}_n \subseteq \mathcal{V}_n$, so $x \in \mathcal{V}_n$, showing that Δ is a countable subcover of $\mathcal{O}$, which is a contradiction. Hence, $(X,\,\tau)$ is Lindelöf. □

Theorem 1.12. *If $(X,\,\tau)$ is second countable and sequentially compact, then it is compact.*

Proof. Second countable implies Lindelöf and sequentially compact implies countably compact, so $(X,\,\tau)$ is a countably compact Lindelöf space, and is therefore compact. □

Theorem 1.13. *If $(X,\,\tau)$ is metrizable, then it is compact if and only if it is countably compact.*

Proof. Compact always implies countably compact. Let's reverse this. Metrizable spaces are first-countable, and hence sequential, so if (X, τ) is countably compact, it is sequentially compact by a previous theorem. By the equivalence of compactness theorem, metrizable spaces are sequentially compact if and only if they are compact. So (X, τ) is compact. $\square$

Theorem 1.14. *If (X, τ) is countably compact, and if $f : X \to \mathbb{R}$ is continuous with respect to the standard topology $\tau_{\mathbb{R}}$, then f is bounded.*

Proof. For if not, if f is unbounded, then for all $n \in \mathbb{N}$, since f is continuous $f^{-1}\big[(-n,\, n)\big]$ is an open subset of X and the set:

$$\mathcal{O} = \Big\{ f^{-1}\big[(-n,\, n)\big] \mid n \in \mathbb{N} \Big\} \tag{12}$$

is a countable open cover of X that has no finite subcover since f is unbounded, which is a contradiction since f is countably compact. Hence, f is bounded. $\square$

Theorem 1.15. *If (X, τ_X) is countably compact, if (Y, τ_Y) is a topological space, and if $f : X \to Y$ is a continuous function, then $(f[X],\, \tau_{Y_{f[X]}})$ is countably compact, where $\tau_{Y_{f[X]}}$ is the subspace topology.*

Proof. The proof is a mimicry of the idea for compact spaces. Given a countable open cover of $f[X]$, since f is continuous this pulls back to a countable cover of X. Since (X, τ_X) is countably compact there is a finite subcover, which pushes forward to a finite subcover of $f[X]$. $\square$

Theorem 1.16 (Generalized Extreme Value Theorem). *If (X, τ) is countably compact, and if $f : X \to \mathbb{R}$ is continuous with respect to the standard topology $\tau_{\mathbb{R}}$, then there are points $x_{min}, x_{max} \in X$ such that for all $x \in X$, $f(x_{min}) \le f(x) \le f(x_{max})$.*

Proof. Since f is continuous and (X, τ) is countably compact, $f[X] \subseteq \mathbb{R}$ is countably compact. But the real line is metrizable, and subspaces of metrizable spaces are metrizable, meaning $f[X]$ is countably compact and metrizable, which means it is compact. But then by the Heine-Borel theorem, $f[X]$ is closed and bounded. Since it is bounded, there exists $m, M \in \mathbb{R}$ such that m is the infimum, and M is the supremum. Since $f[X]$ is closed, $m, M \in f[X]$, meaning there are $x_{\text{min}}, x_{\text{max}}$ such that $f(x_{\text{min}}) = m$ and $f(x_{\text{max}}) = M$. Since m and M are the bounds of $f[X]$, for all $x \in X$ we have $f(x_{\text{min}}) \le f(x) \le f(x_{\text{max}})$. $\square$

Theorem 1.17 (Extreme Value Theorem). *If (X, τ) is compact, and if $f : X \to \mathbb{R}$ is continuous with respect to the standard topology $\tau_{\mathbb{R}}$, then there are points $x_{min}, x_{max} \in X$ such that for all $x \in X$, $f(x_{min}) \le f(x) \le f(x_{max})$.*

Proof. Since compact implies countably compact, this follows from the previous theorem. $\square$

This idea is useful enough that it gets a name.

Definition 1.4 (**Pseudocompact Topological Space**) A pseudocompact space is a topological space (X, τ) such that every continuous function $f : X \to \mathbb{R}$ is bounded. ■

The extreme value theorem shows that compact implies pseudocompact. So does sequentially compact. The proof is identical if we know that the continuous image of a sequentially compact space is sequentially compact. Let's prove this.

Theorem 1.18. *If (X, τ_X) is sequentially compact, if (Y, τ_Y) is a topological space, and if $f : X \to Y$ is continuous, then $(f[X], \tau_{Y_{f[X]}})$ is sequentially compact.*

Proof. For suppose not and let $b : \mathbb{N} \to f[X]$ be a sequence with no converent subsequence. Since $b_n \in f[X]$, by the axiom of choice we can find a sequence $a : \mathbb{N} \to X$ such that $f(a_n) = b_n$ for all $n \in \mathbb{N}$. But (X, τ_X) is sequentially compact, so there is a convergent subsequence a_k with limit $x \in X$. That is, $a_{k_n} \to x$. But f is continuous, so $f(a_{k_n}) \to f(x)$. But then $b_{k_n} \to f(x)$, meaning b_k is a converegent subsequence, which is a contradiction. So $(f[X], \tau_{Y_{f[X]}})$ is sequentially compact. □

Theorem 1.19. *If (X, τ) is sequentially compact, then it is pseudocompact.*

Proof. For if $f : X \to \mathbb{R}$ is continuous, then $f[X] \subseteq \mathbb{R}$ is sequentially compact, and since $(\mathbb{R}, \tau_{\mathbb{R}})$ is metrizable, sequentially compact implies compact, meaning $f[X]$ is closed and bounded by the Heine-Borel theorem. Hence, (X, τ) is pseudocompact. □

Theorem 1.20. *If (X, τ) is metrizable, then (X, τ) is compact if and only if it is pseudocompact.*

Proof. Compact always implies pseudocompact. Since (X, τ) is metrizable, to prove pseudocompact implies compact it is sufficient to prove that (X, τ) is sequentially compact (since compact and sequentially compact are equivalent in metrizable spaces). Suppose (X, τ) is not sequentially compact, and let d be a metric that induces τ. Then there is a sequence $a : \mathbb{N} \to X$ with no convergent subsequence. Then the set $A \subseteq X$ defined by:

$$A = \{\, a_n \in X \mid n \in \mathbb{N} \,\} \tag{13}$$

is closed. Not only that, but (A, τ_A) as a subspace is discrete. For given $x \in X$ there is some $\varepsilon_x > 0$ such that the ε_x ball around x contains at most one element of A (otherwise we could obtain a convergent subsequence tending to x). In particular, we can apply this to every $a_n \in A$ meaning there is an open subset in A that contains only a_n. Using this, define $f : A \to \mathbb{R}$ via $f(a_n) = n$. Then, since A is closed, and since f is continuous since A is a discrete space, by the Tietze extension theorem there is a continuous function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}|_A = f$. But $\tilde{f}$ is not bounded, which is a contradiction since (X, τ) is pseudocompact. Hence, (X, τ) is sequentially compact, and therefore compact since (X, τ) is metrizable. □

The real line has the property that it can be written as the union of countably many compact sets. Namely, given $n \in \mathbb{N}$, define $\mathcal{C}_n = [-n, n]$. Then each $\mathcal{C}_n$ is a compact subset and $\bigcup_n \mathcal{C}_n = \mathbb{R}$, so $\mathbb{R}$ is the union of countably many compact sets. This gets a name.

Definition 1.5 (**σ Compact Topological Space**) A σ compact topological space is a topological space (X, τ) such that there exists a countable set $\mathcal{O}$ such that for all $\mathcal{C} \in \mathcal{O}$, $\mathcal{C} \subseteq X$ is compact, and such that $X = \bigcup \mathcal{O}$. ■

The real line, complex plane (or Euclidean plane), and Euclidean space are all Lindelöf spaces, even though they are not compact. This has some use in analysis. It's a lot easier to see that $\mathbb{R}^n$ is σ compact since closed balls of radius n for all $n \in \mathbb{N}$ create a countable collection of compact subsets that cover the space. Fortunately, σ compact implies the Lindelöf property.

Theorem 1.21. *If (X, τ) is σ compact, then it is Lindelöf.*

Proof. For if not, then there is an open cover $\mathcal{O} \subseteq \tau$ with no countable subcover. But (X, τ) is σ compact so there is a countable set $\mathcal{B}$ of compact subsets of X such that $X = \bigcup \mathcal{B}$. Since $\mathcal{B}$ is countable there is a surjection $A : \mathbb{N} \to \mathcal{B}$. But then for all $n \in \mathbb{N}$, (A_n, τ_{A_n}) is compact, by hypothesis, and $\mathcal{O}$ covers A_n. So there is a finite subcover Δ_n. But then, since Δ_n is finite for all $n \in \mathbb{N}$, the set $\Delta \subseteq \mathcal{O}$ defined by:

$$\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n \tag{14}$$

is countable. But Δ is a cover of X since given $x \in X$ there is an $n \in \mathbb{N}$ such that $x \in A_n$, and hence $x \in \bigcup \Delta_n$, but $\Delta_n \subseteq \Delta$, so $x \in \bigcup \Delta$. So Δ is a countable subcover of $\mathcal{O}$, a contradiction. Hence, (X, τ) is Lindelöf. □

There is one more property that is ever-so-slightly stronger than σ compact, but has found quite a lot of use in the theory of manifolds and Riemannian geometry. The idea of being *compactly exhaustible.*

Definition 1.6 (**Compactly Exhaustible Topological Space**) A compactly exhaustible topological space is a topological space (X, τ) such that there is a sequence $A : \mathbb{N} \to \mathcal{P}(X)$ such that for all $n \in \mathbb{N}$ it is true that A_n is a compact subset, $A_n \subseteq \mathrm{Int}_\tau(A_{n+1})$, and such that $X = \bigcup_{n \in \mathbb{N}} A_n$. ■

Euclidean space is compactly exhaustible (every manifold is). Take $\mathcal{C}_n$ to be the closed ball of radius n. Then, just like with σ compact, these sets are all compact and cover $\mathbb{R}^n$, but also the closed ball of radius n fits entirely inside the open ball of radius $n+1$ showing that $\mathcal{C}_n \subseteq \mathrm{Int}_\tau(\mathcal{C}_{n+1})$. Now, for the result. Compactly exhaustible implies σ compact.

Theorem 1.22. *If (X, τ) is compactly exhaustible, then it is σ compact.*

Proof. Let $A : \mathbb{N} \to \mathcal{P}(X)$ be a sequence such that A_n is compact, $A_n \subseteq \mathrm{Int}_\tau(A_{n+1})$, and $\bigcup_{n \in \mathbb{N}} A_n = X$. Then the set:

$$\mathcal{O} = \{\, A_n \mid n \in \mathbb{N} \,\} \tag{15}$$

is a countable collection of subsets that are compact and cover X, so (X, τ) is σ compact. $\square$