Point-Set Topology: Lecture 28

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1 Compactifications

Compact topological spaces are nice. Unlike the Hausdorff and first countable conditions, which most of your every day spaces possess, compact is not as common. Compact spaces are just too convenient. For example $\mathbb{R}^n$ is not compact for all $n > 0$ by the Heine-Borel theorem. A *compactification* of a topological space (X, τ) is a compact space $(\tilde{X}, \tilde{\tau})$ that contains (X, τ) is an embedded subspace. There are two common compactifications that are used in topology and analysis, the one point compactification, and the Stone-Čech compactification.

The one point compactification, also called the Alexandroff extension or the Alexandroff compactification, takes (X, τ) and adds one point in a way that makes it compact. This new point is often denoted ∞, but what if X already had ∞ as an element? Is there a way to guarantee we are adding a new element to X that is not already contained in it? The axioms of set theory tell us if A is a set, then $A \notin A$, so we can form our new set $\tilde{A}$ via $A \cup \{ A \}$.

Definition 1.1 (**One Point Compactification**) The one point compactification of a topological space (X, τ) is the ordered pair $(\tilde{X}, \tilde{\tau})$ where $\tilde{X} = X \cup \{ X \}$ and $\tilde{\tau}$ is defined by:

$$\tilde{\tau} = \big\{ \mathcal{U} \subseteq \tilde{X} \mid \mathcal{U} \in \tau \text{ or } \mathcal{U} = (X \setminus \mathcal{C}) \cup \{ X \}, \mathcal{C} \text{ compact and closed.} \big\} \tag{1}$$

The new element $\{ X \}$ that is added is often denoted ∞ (if we know the symbol ∞ does not already belong to X). The set $\tilde{\tau}$ is thus the set of all open sets in τ plus all complements of closed compact sets together with infinity. ■

There are several theorems related to the one point compactification that we lack the time to go over, so I'll just present them.

- The one point compactification of a topological space is a topological space (that is, $\tilde{\tau}$ is a topology on $\tilde{X}$).
- (X, τ) is a subspace of $(\tilde{X}, \tilde{\tau})$ and the inclusion map $\iota : X \to \tilde{X}$ is an embedding.

- The one point compactification of a topological space is compact.
- If (X, τ) is a non-compact topological space, then it is a dense subspace of its one point compactification.
- The one point compactification of a topological space (X, τ) is Hausdorff if and only if (X, τ) is locally compact and Hausdorff. The one point compactification of $\mathbb{Q}$ is an example of a non-Hausdorff compactification, even though $\mathbb{Q}$ is Hausdorff (it is not locally compact, however).
- The one point compactification of a topological space (X, τ) is Fréchet if and only if (X, τ) is Fréchet.

The one point compactification makes the following theorem a lot easier.

Theorem 1.1. *If (X, τ) is locally compact and Hausdorff, then it is regular.*

Proof. Let $(\tilde{X}, \tilde{\tau})$ be the one point compactification of (X, τ). Since (X, τ) is locally compact and Hausdorff, $(\tilde{X}, \tilde{\tau})$ is compact and Hausdorff. But then $(\tilde{X}, \tilde{\tau})$ is regular. But (X, τ) is a subspace of $(\tilde{X}, \tilde{\tau})$, and a subspace of a regular space is regular. Hence, (X, τ) is regular. □

The Stone-Cĕch compactification is another common tool used in analysis and topology, but its description is a lot harder to convey. The simplest definition is the *spectrum* of the set $C_b(X, \mathbb{C})$ of bounded continuous functions from a topological space (X, τ) into the complex numbers $\mathbb{C}$, which is equipped with the standard Euclidean topology of $\mathbb{R}^2$. A discussion of this idea would require an excursion into analysis that we don't have the time for, unfortunately. I will mention that the Stone-Čech compactification allows one to describe *completely metrizable* spaces.

Definition 1.2 (**Completely Metrizable Topological Space**) A completely metrizable topological space is a topological space (X, τ) such that there is a complete metric d on X that induces τ. ■

Theorem 1.2. *If (X, τ) is a metrizable topological space, then it is completely metrizable if and only if it is a G_δ subspace of its Stone-Čech compactification.*

Lastly, Stone spaces are compact totally disconnected Hausdorff topological spaces. They arise in the study of the Stone-Čech compactification, but their use is far broader. Every Boolean algebra $(B, \wedge, \vee)$ is represented by a Stone space, and conversely every Stone space represents a Boolean algebra. Boolean algebras capture the algebraic structure of logic. Think of propositions together with *and* and *or*. Boolean algebras satisfy, for all $P, Q, R \in B$, the following:

$$P \wedge Q = Q \wedge P \qquad P \vee Q = Q \vee P \tag{2}$$

$$P \wedge (Q \wedge R) = (P \wedge Q) \wedge R \qquad P \vee (Q \vee R) = (P \vee Q) \vee R \tag{3}$$

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \qquad P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R) \tag{4}$$

$$P \vee \text{False} = P \qquad P \wedge \text{True} = P \tag{5}$$

$$P \vee \neg P = \text{True} \qquad P \wedge \neg P = \text{False} \tag{6}$$

So amazingly topology and logic become the same study.