Point-Set Topology: Lecture 29

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1 Locally Euclidean Topological Spaces

We now reach the final topic of the course, *manifolds*. More general than manifolds, we start with locally Euclidean spaces.

Definition 1.1 (**Locally Euclidean Topological Space**) A locally Euclidean topological space is a topological space $(X,\,\tau)$ such that for all $x\in X$ there is a $\mathcal{U}\in\tau$ an $n\in\mathbb{N}$, and an injective continuous open mapping $\varphi:\mathcal{U}\to\mathbb{R}^n$ such that $x\in\mathcal{U}$. ■

Example 1.1 For all $n\in\mathbb{N}$ the Euclidean space $\mathbb{R}^n$ with the Euclidean topology is locally Euclidean. For all $\mathbf{x}\in\mathbb{R}^n$ choose $\mathcal{U}=\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^n$ to be the identity mapping $f(\mathbf{x})=\mathrm{id}_{\mathbb{R}^n}(\mathbf{x})=\mathbf{x}$. ■

Example 1.2 If $\mathcal{U}\subseteq\mathbb{R}^n$ is an open subset with respect to the Euclidean topology, then $(\mathcal{U},\,\tau_{\mathbb{R}^n_{\mathcal{U}}})$ is locally Euclidean. Given $x\in\mathcal{U}$ define $f:\mathcal{U}\to\mathbb{R}^n$ via $f=\mathrm{id}_{\mathbb{R}^n}|_{\mathcal{U}}$. Then f is injective and continuous, and since $\mathcal{U}$ is open, f is also an open mapping. ■

Example 1.3 The solution to $y^2-x^2=0$ in the plane is not locally Euclidean. This forms an $\mathbf{X}$, $y=\pm|x|$. Every point except the origin is locally Euclidean, locally looking like $\mathbb{R}$. The origin is where this goes wrong. No matter how much you zoom in it still locally looks like an $\mathbf{X}$. This is certainly not locally like $\mathbb{R}$, but it's also not 2 dimensional. Similarly, it's not n dimensional for any $n\in\mathbb{N}$. Thus this subspace of $\mathbb{R}^2$ is not locally Euclidean. ■

This example tells us that closed subspaces of locally Euclidean spaces do not need to be locally Euclidean.

Example 1.4 The bug-eyed line is locally Euclidean, second countable, but not Hausdorff. Every point other than the two origins is locally like $\mathbb{R}$. The two origins are also locally like $\mathbb{R}$. See Fig. 1. ■

Example 1.5 The branching line is another example of a non-Hausdorff space that is locally Euclidean. The construction is similar to the bug-eyed line. Take $X\subseteq\mathbb{R}^2$ to be the set of all points of the form $(x,\,y)\in\mathbb{R}^2$ such that $x\in\mathbb{R}$ and $y=\pm1$. Define $(x_0,\,y_0)R(x_1,\,y_1)$ if and only if $x_0=x_1$ and $x_0<0$. The

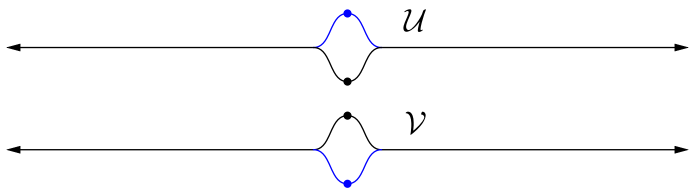

Figure 1: The Bug-Eyed Line

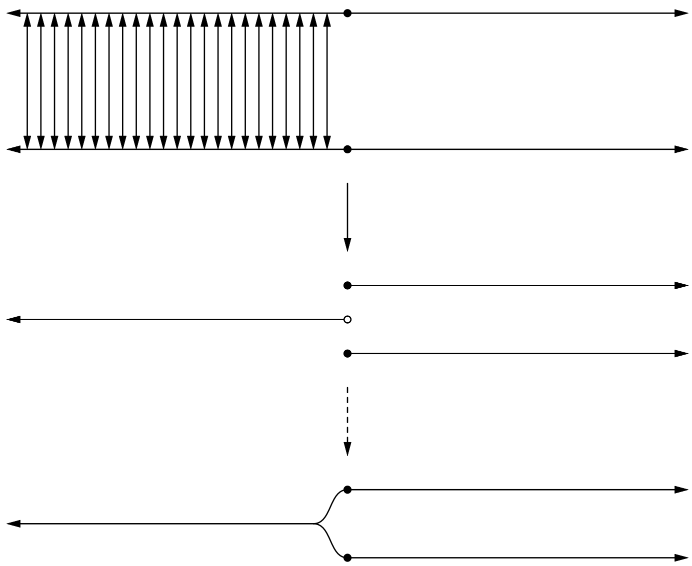
Figure 2: The Branching Line Construction

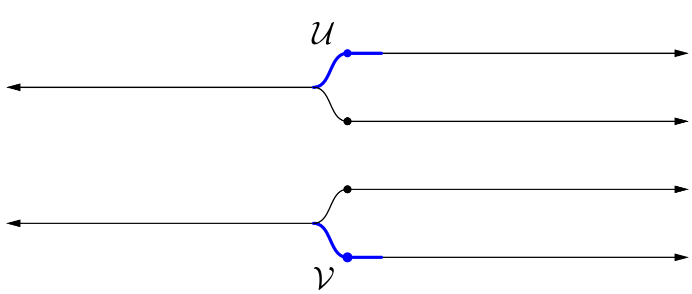

Figure 3: The Branching Line is Locally Euclidean

branching line is the quotient X/R with the quotient topology (See Fig. 2). Like the bug-eyed line, it too is locally Euclidean, see Fig. 3. ■

Example 1.6 The long line is locally Euclidean and Hausdorff, but not second countable. It is also not paracompact. ■

Example 1.7 As far as set theory is concerned, a function $f : A \to B$ from a set A to a set B is a subset of $A \times B$ satisfying certain properties. We can use this to define locally Euclidean topological spaces by looking at continuous functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ for some $m, n \in \mathbb{N}$. Given $f : \mathbb{R}^m \to \mathbb{R}^n$, continuous, $f \subseteq \mathbb{R}^m \times \mathbb{R}^n$ can be given the subspace topology. This makes it a closed subset since f is continuous. It is also a locally Euclidean subspace. For given $\big(\mathbf{x},\, f(\mathbf{x})\big) \in f$, let $\mathcal{U} = f$ and define $F : f \to \mathbb{R}^m$ via:

$$F\big((\mathbf{x},\, f(\mathbf{x})\big) = \mathbf{x} \tag{1}$$

This is just the projection of the elements of $f \subseteq \mathbb{R}^m \times \mathbb{R}^n$ onto $\mathbb{R}^m$. Projections are continuous. Let's show F is injective and an open mapping. It is injective since given:

$$\big(\mathbf{x}_0,\, f(\mathbf{x}_0)\big) \neq \big(\mathbf{x}_1,\, f(\mathbf{x}_1)\big) \tag{2}$$

we must have $\mathbf{x}_0 \neq \mathbf{x}_1$ (since if $\mathbf{x}_0 = \mathbf{x}_1$, then $f(\mathbf{x}_0) = f(\mathbf{x}_1)$ by definition of a function). So then:

$$F\big((\mathbf{x}_0,\, f(\mathbf{x}_0)\big) \neq F\big((\mathbf{x}_1,\, f(\mathbf{x}_1)\big) \tag{3}$$

meaning F is injective. There is a continuous inverse $F^{-1} : \mathbb{R}^m \to f$ given by:

$$F^{-1}(\mathbf{x}) = \big(\mathbf{x},\, f(\mathbf{x})\big) \tag{4}$$

Since f is continuous, F^{-1} is continuous since both components are continuous. So F is an open mapping and f is a locally Euclidean subspace of $\mathbb{R}^m \times \mathbb{R}^n$. ■

Example 1.8 $\mathbb{S}^1$ with the subspace topology from $\mathbb{R}^2$ is locally Euclidean. We'll show this in two ways. First, via orthographic projection. We split the circle into four parts:

$$\mathcal{U}_{\text{North}} = \{\, (x,\, y) \in \mathbb{S}^1 \mid y > 0 \,\} \tag{5}$$
$$\mathcal{U}_{\text{South}} = \{\, (x,\, y) \in \mathbb{S}^1 \mid y < 0 \,\} \tag{6}$$
$$\mathcal{U}_{\text{East}} = \{\, (x,\, y) \in \mathbb{S}^1 \mid x > 0 \,\} \tag{7}$$
$$\mathcal{U}_{\text{West}} = \{\, (x,\, y) \in \mathbb{S}^1 \mid x < 0 \,\} \tag{8}$$

See Fig. 4. Then we define four functions:

$$\varphi_{\text{North}} : \mathcal{U}_{\text{North}} \to \mathbb{R} \qquad \varphi_{\text{North}}\big((x,\, y)\big) = x \tag{9}$$
$$\varphi_{\text{South}} : \mathcal{U}_{\text{South}} \to \mathbb{R} \qquad \varphi_{\text{South}}\big((x,\, y)\big) = x \tag{10}$$
$$\varphi_{\text{East}} : \mathcal{U}_{\text{East}} \to \mathbb{R} \qquad \varphi_{\text{East}}\big((x,\, y)\big) = y \tag{11}$$
$$\varphi_{\text{West}} : \mathcal{U}_{\text{West}} \to \mathbb{R} \qquad \varphi_{\text{West}}\big((x,\, y)\big) = y \tag{12}$$

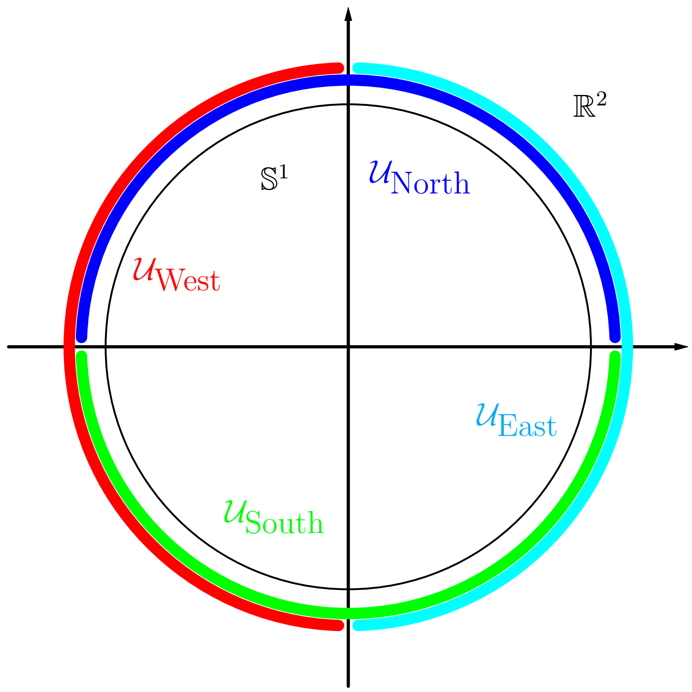

Figure 4: Cover of $\mathbb{S}^1$ with Locally Euclidean Sets

Since these are projection mappings, they are continuous. From how the four open sets are defined, each is also injective. To show it is an open mapping we just need to find a continuous inverse with respect to the image of these sets. Note that for all four functions the range of $(-1,\,1)$. We have the following inverse functions:

$$\varphi_{\text{North}}^{-1}(x) = \big(x,\, \sqrt{1-x^2}\big) \tag{13}$$

$$\varphi_{\text{South}}^{-1}(x) = \big(x,\, -\sqrt{1-x^2}\big) \tag{14}$$

$$\varphi_{\text{East}}^{-1}(y) = \big(\sqrt{1-y^2},\, y\big) \tag{15}$$

$$\varphi_{\text{West}}^{-1}(y) = \big(-\sqrt{1-y^2},\, y\big) \tag{16}$$

each of which is continuous since the square root function is continuous. The four sets also cover $\mathbb{S}^1$, showing that $\mathbb{S}^1$ is locally Euclidean. ■

This shows we can cover $\mathbb{S}^1$ using four sets each of which is homeomorphic to an open subset of $\mathbb{R}$. We can do better, only two sets are needed. Place an observer at the north pole $N = (0,\,1)$. Given any other point $(x,\,y)$ the line from the observer to the point is not parallel to the x axis, meaning eventually it must intersect it. Let's solve for when. The line segment $\alpha(t) = (1-t)N + t(x,\,y)$ starts at the north pole at time $t = 0$ and ends at the point $(x,\,y)$ on the circle at time $t = 1$. The line intersects the x axis when the y component is zero.

Thus we wish to solve $1 - t + ty = 0$ for t. We get:

$$t_0 = \frac{1}{1 - y} \tag{17}$$

The x coordinate at time $t = t_0$ is then:

$$\varphi_N\big((x,\, y)\big) = \frac{x}{1 - y} \tag{18}$$

This is *stereographic projection* about the north pole. It is continuous since it is a rational function. It is also bijective with a continuous inverse. Given $X \in \mathbb{R}$ we can solve for the value $(x,\, y) \in \mathbb{S}^1$ that gets mapped to X by reversing the previous process. The line $\beta(t) = (1 - t)N + t(X,\, 0)$ starts at the north pole and ends at $(X,\, 0)$. We wish to solve for the time t when $||\beta(t)||_2 = 1$ which corresponds to the moment the line intersects the circle. We have:

$$\begin{aligned}
||\beta(t)||_2 &= ||(1 - t)N + t(X,\, 0)||_2 && (19)\\
&= ||(1 - t)(0,\, 1) + t(X,\, 0)||_2 && (20)\\
&= ||(tX,\, 1 - t)||_2 && (21)\\
&= \sqrt{(tX)^2 + (1 - t)^2} && (22)
\end{aligned}$$

Solving for $||\beta(t)||_2 = 1$ is equivalent to solving $||\beta(t)||_2^2 = 1$ so we need to consider the expression $(tX)^2 + (1 - t)^2$. We get:

$$\begin{aligned}
1 &= (tX)^2 + (1 - t)^2 && (23)\\
&= t^2X^2 + 1 - 2t + t^2 && (24)\\
&= t^2(1 + X^2) - 2t + 1 && (25)
\end{aligned}$$

meaning we want to solve for $t^2(1+X^2)-2t = 0$. The solution $t = 0$ corresponds to the North pole, which is not the one we want. Dividing through by t we get:

$$t_1 = \frac{2}{1 + X^2} \tag{26}$$

The point $(x,\, y)$ corresponds to $\beta(t_1)$ and is given by:

$$\varphi_N^{-1}(X) = \Big(\frac{2X}{1 + X^2},\, \frac{-1 + X^2}{1 + X^2}\Big) \tag{27}$$

This function is continuous since it is a rational function in each component. Because of this $\varphi_N : \mathbb{S}^1 \setminus \{\, (0,\, 1)\,\} \to \mathbb{R}$ is a homeomorphism. Doing a similar projection about the south pole shows that $\mathbb{S}^1$ can be covered by two open sets, $\mathbb{S}^1 \setminus \{\, (0,\, 1)\,\}$ and $\mathbb{S}^1 \setminus \{\, (0,\, -1)\,\}$, each of which is homeomorphic to $\mathbb{R}$.

It is impossible to do this with one set. This is because $\mathbb{S}^1$ is not homeomorphic to an open subset of $\mathbb{R}$ since $\mathbb{S}^1$ is compact and the only open subset of $\mathbb{R}$ that is compact is the empty set, but $\mathbb{S}^1$ is non-empty. So two is the best we can do.

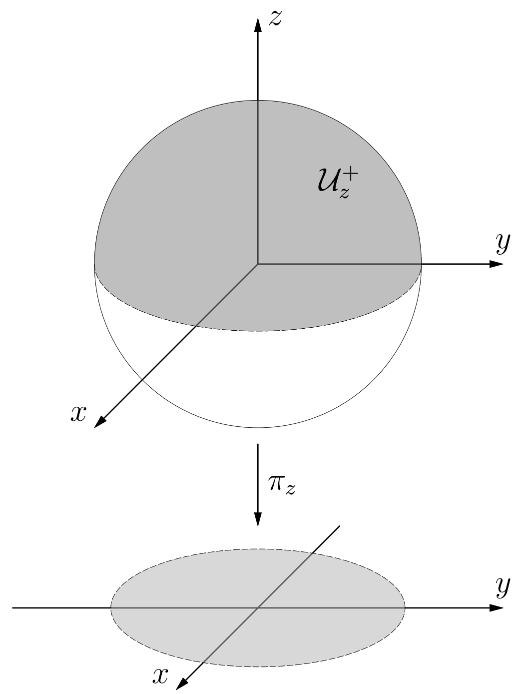

Figure 5: Orthographic Projection of the Sphere

Example 1.9 The sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is also locally Euclidean for all $n \in \mathbb{N}$. Define $\mathcal{U}_k^{\pm} \subseteq \mathbb{S}^n$ via:

$$\mathcal{U}_k^+ = \{\, \mathbf{x} \in \mathbb{S}^n \mid \mathbf{x}_k > 0 \,\} \tag{28}$$

$$\mathcal{U}_k^- = \{\, \mathbf{x} \in \mathbb{S}^n \mid \mathbf{x}_k < 0 \,\} \tag{29}$$

These $2n+2$ open sets cover $\mathbb{S}^n$ and each is homeomorphic to an open subset of $\mathbb{R}^n$. Define $\varphi_k^{\pm} : \mathcal{U}_k^{\pm} \to B_1^{\mathbb{R}^n}(\mathbf{0})$ via:

$$\varphi_k^{\pm}(\mathbf{x}) = (\mathbf{x}_0,\, \dots,\, \mathbf{x}_{k-1},\, \mathbf{x}_{k+1},\, \mathbf{x}_n) \tag{30}$$

That is, projecting down that k^{th} axis. This is continuous with a continuous inverse ${\varphi_k^{\pm}}^{-1} : B_1^{\mathbb{R}^n}(\mathbf{0}) \to \mathcal{U}_k^{\pm}$ given by:

$${\varphi_k^{\pm}}^{-1}(\mathbf{x}) = (\mathbf{x}_0,\, \dots,\, \mathbf{x}_{k-1},\, \pm\sqrt{1 - ||\mathbf{x}||_2^2},\, \mathbf{x}_k,\, \dots,\, \mathbf{x}_{n-1}) \tag{31}$$

This is also continuous, so $\mathbb{S}^n$ is locally Euclidean. ■

These mappings are called *orthographic projections*. They are formed by placing an observer at *infinity* and projecting what they see down to the plane. This is shown in Fig. 5

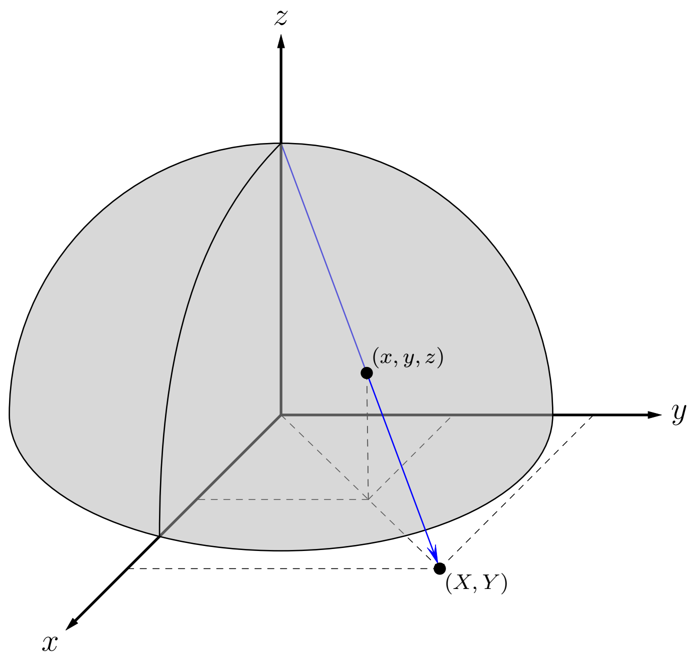

Figure 6: Stereographic Projection for the Sphere

Definition 1.2 (**Topological Chart**) A topological chart of dimension n in a topological space $(X,\,\tau)$ about a point $x\in X$ is an ordered pair $(\mathcal{U},\,\varphi)$ such that $\mathcal{U}\in\tau$, $x\in\mathcal{U}$, and $\varphi:\mathcal{U}\rightarrow\mathbb{R}^n$ is an injective continuous open mapping. ■

Locally Euclidean could equivalently be described as a topological space $(X,\,\tau)$ such that for all $x\in X$ there is a chart $(\mathcal{U},\,\varphi)$ such that $x\in\mathcal{U}$. A collection of charts that covers a space is called an *atlas*.

Definition 1.3 (**Topological Atlas**) A topological atlas for a topological space $(X,\,\tau)$ is a set $\mathcal{A}$ of topological charts in $(X,\,\tau)$ such that for all $x\in X$ there is a $(\mathcal{U},\,\varphi)\in\mathcal{A}$ such that $x\in\mathcal{U}$. ■

That is, an atlas is a collection of charts whose domains cover the space. Think of an actual atlas used for navigating. The pages consist of various locations on the globe, but only provides local information. To get information that is more global requires piecing some of the charts of the atlas together. A locally Euclidean space is a topological space $(X,\,\tau)$ such that there exists an atlas $\mathcal{A}$ for it. We've shown that $\mathbb{S}^n$ can be covered by $2n+2$ charts using orthographic projection. We can do better using stereographic projection the same way we did for $\mathbb{S}^1$. This is shown for $\mathbb{S}^2$ in Fig. 6.

There are two other types of projections that are useful for geometric reasons in covering $\mathbb{S}^n$. These are the near-sided and far-sided projections. Near-sided projection is shown in Fig. 7. The idea is to take an observer and place them

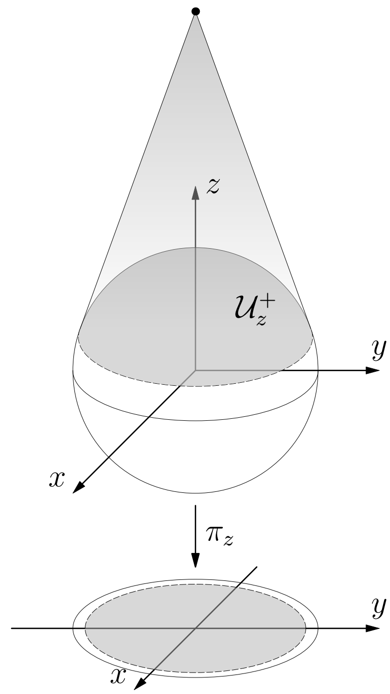

Figure 7: Near-Sided Projection of the Sphere

somewhere on the z axis above the sphere. The portion of the sphere that is visible is then projected down to the xy plane. Far-sided projection is the opposite. You place the observer at the same spot but remove everything that can be seen. The result is a hollow semi-sphere. You then unwrap this on to the plane to get the projection. This is shown in Fig. 8. Stereographic projection is then just far-sided projection at the north pole, and orthographic projection is far-sided projection at infinity.

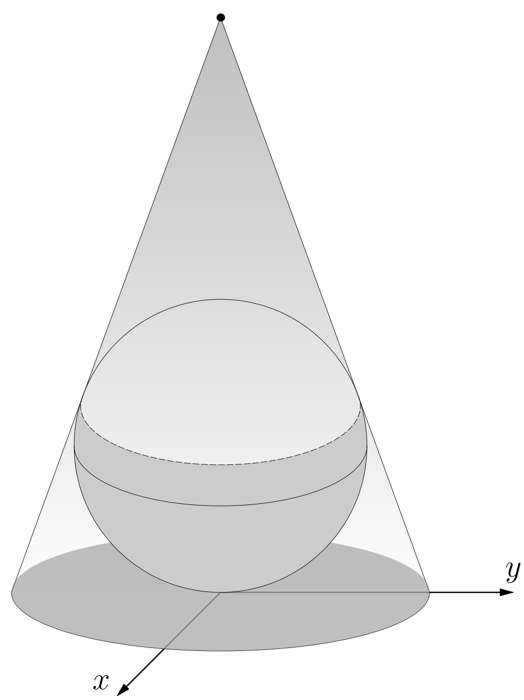

Figure 8: Far-Sided Projection of the Sphere