Point-Set Topology: Lecture 30

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1 Manifolds

So far we've seen several examples of locally Euclidean spaces.

- $\mathbb{R}^n$ for all $n \in \mathbb{N}$.
- Open subspaces of $\mathbb{R}^n$.
- The n sphere $\mathbb{S}^n$
- The graphs of continuous functions $f : \mathbb{R}^m \to \mathbb{R}^n$ with the subspace topology in $\mathbb{R}^m \times \mathbb{R}^n$.
- The bug-eyed line.
- The branching line.
- The long line.

The first four are subspaces of Euclidean space, the last three are not. We know these last three spaces cannot be embedded into $\mathbb{R}^n$ since the bug-eyed and branching lines are not Hausdorff and all subspaces of Hausdorff spaces are Hausdorff, and the long line is not second countable. Topological manifolds add the Hausdorff property and second countability to ensure nothing too weird can happen with the space.

Definition 1.1 (**Topological Manifold**) A topological manifold is a topological space $(X,\,\tau)$ that is Hausdorff, second countable, and locally Euclidean. That is, for all distinct $x, y \in X$ there are open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. There is also a countable basis $\mathcal{B}$ for the topology τ. And for all $x \in X$ there is a topological chart $(\mathcal{U},\,\varphi)$ in $(X,\,\tau)$ such that $x \in \mathcal{U}$. ■

The word *topological* is added to manifold since there are several types of manifolds. In many cases the adjective is dropped and context is required to know which type of manifold is being talked about. A few of the other types of manifold are listed below.

- Topological manifold: A type of topological space.

- Smooth manifold: A topological manifold with extra structure so that it is possible to ask if functions are differentiable and to speak of things like tangent vectors and vector fields.
- Riemannian manifold: A smooth manifold that is equipped with a method of assigning angles between tangent vectors, measuring lengths of curves, and the area and volume of subspaces.
- Lorentz Manifold: A smooth manifold with, loosely speaking, a method of differentiating between time and space. Lorentz manifolds fall into the study of spacetime and general relativity.
- Semi-Riemannian manifold: A generalization of Lorentz and Riemannian manifolds. In particular, all Lorentz and all Riemannian manifolds are also semi-Riemannian manifolds.

We will talk about smooth manifolds briefly, since we actually have all of the terminology to discuss them. We won't dive too deep into the field however since smooth manifolds belong to differential topology. Riemannian and Lorentz manifolds belong to geometry, and we won't have anything to say about those (we also lack the algebraic terminology required to define them).

Example 1.1 Euclidean space $\mathbb{R}^n$ is a topological manifold. It is locally Euclidean from the previous lecture, and it is also Hausdorff and second countable. ■

Example 1.2 Any open subset $\mathcal{U} \subseteq \mathbb{R}^n$ with the subspace topology is a topological manifold. It is locally Euclidean from the previous lecture, and since subspaces of second countable Hausdorff spaces are still second countable and Hausdorff, $\mathcal{U}$ is a topological manifold. ■

Example 1.3 The n dimensional sphere $\mathbb{S}^n$ is a topological manifold. We used orthographic, stereographic, near-sided, and far-sided projections last lecture to show $\mathbb{S}^n$ is locally Euclidean in several different ways. Since the n sphere is a subspace of $\mathbb{R}^{n+1}$, it is second countable and Hausdorff. ■

Example 1.4 The long line is not a topological manifold since it is not second countable. It is locally Euclidean and Hausdorff, however. ■

Example 1.5 The bug-eyed line is not a topological manifold. It is second countable and locally Euclidean, but it is not Hausdorff. ■

Example 1.6 Similarly, the branching line is not a topological manifold since it is not Hausdorff. ■

I would like to think the real reason for the Hausdorff and second countability requirements is so that we can perhaps hope that topological manifolds are really just particular subspaces of $\mathbb{R}^n$. From the definition there is no such requirement and topological manifolds can be considered as abstract objects that do not live in any ambient Euclidean space. This can be quite useful. The spacetime of general relativity is a four dimensional topological manifold (it's actually a Lorentz manifold, but let's not dive into that). Whether or not it is possible to embed spacetime into some higher dimensional Euclidean space or not seems irrelevant to any physical problem one might study. For the curious, it is indeed possible to embed a four dimensional spacetime into $\mathbb{R}^8$. What purpose eight dimensional Euclidean space may serve for any physics problems is beyond me.

We now arrive at our first set of spaces that are not obviously some subspace of $\mathbb{R}^n$. These are the *real projective spaces* and are denoted $\mathbb{RP}^n$.

Example 1.7 (**Real Projective Space**) Let $X = \mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\}$. Define the equivalence relation R on X via $\mathbf{x}R\mathbf{y}$ if and only if $\mathbf{y} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}\setminus\{\,0\,\}$. $\mathbb{RP}^n$ is the set X/R and the topology $\tau_{\mathbb{RP}^n}$ is the quotient topology induced by R. As a set this is the set of all lines in $\mathbb{R}^{n+1}$ that pass through the origin. That is, a point $[\mathbf{x}] \in \mathbb{RP}^n$ is the entire line through the origin that passes through the point $\mathbf{x}$. Let's start with $\mathbb{RP}^1$. Any line can be described by an angle $0 \le \theta < \pi$. If you vary the line you are on slightly, you are just varying this angle. Hopefully it becomes intuitive that $\mathbb{RP}^1$ is in fact a one dimensional locally Euclidean space (it may not be intuitive as to why it is Hausdorff or second countable, but we'll get there). A similar thinking applies to $\mathbb{RP}^n$. Let's be precise. Let $\mathcal{U}_k \subseteq X$ be defined by:

$$\mathcal{U}_k = \{\,\mathbf{x} \in \mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\} \mid \mathbf{x}_k \neq 0\,\} \tag{1}$$

This is the complement of the k^{th} axis, which is open since the k^{th} axis is closed. It is also saturated with respect to the canonical quotient map $q : X \to \mathbb{RP}^n$ defined by $q(\mathbf{x}) = [\mathbf{x}]$. That is, $q^{-1}\big[q[\mathcal{U}_k]\big] = \mathcal{U}_k$. It is always the case that $\mathcal{U}_k \subseteq q^{-1}\big[q[\mathcal{U}_k]\big]$, let's show this reverses for our particular set $\mathcal{U}_k$. Let $\mathbf{x} \in q^{-1}\big[q[\mathcal{U}_k]\big]$. Then $[\mathbf{x}] \in q\big[\mathcal{U}_k\big]$ so there is some $\mathbf{y} \in \mathcal{U}_k$ such that $[\mathbf{x}] = [\mathbf{y}]$. But then $\mathbf{y}_k \neq 0$ and $\mathbf{x} = \lambda\mathbf{y}$ for some $\lambda \in \mathbb{R} \setminus \{\,0\,\}$. But then $\mathbf{x}_k \neq 0$, and hence $\mathbf{x} \in \mathcal{U}_k$. So $\mathcal{U}_k$ is saturated. But since q is a quotient map, if $\mathcal{U}_k$ is open and saturated, the set $\tilde{\mathcal{U}}_k = q[\mathcal{U}_k]$ is open. Define $\varphi_k : \tilde{\mathcal{U}}_k \to \mathbb{R}^n$ via:

$$\varphi_k\big([\mathbf{x}]\big) = \Big(\frac{\mathbf{x}_0}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_{k-1}}{\mathbf{x}_k},\,\frac{\mathbf{x}_{k+1}}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_n}{\mathbf{x}_k}\Big) \tag{2}$$

We have to prove this is well-defined in two regards. First, there is no division by zero since $\mathbf{x} \in \mathcal{U}_k$ implies $\mathbf{x}_k \neq 0$. Second, this is well defined as a function. By that I mean if $[\mathbf{x}] = [\mathbf{y}]$, then there is some $\lambda \in \mathbb{R} \setminus \{\,0\,\}$ such that $\mathbf{y} = \lambda\mathbf{x}$. But then:

$$\begin{aligned}
\varphi_k\big([\mathbf{y}]\big) &= \Big(\frac{\mathbf{y}_0}{\mathbf{y}_k},\,\dots,\,\frac{\mathbf{y}_{k-1}}{\mathbf{y}_k},\,\frac{\mathbf{y}_{k+1}}{\mathbf{y}_k},\,\dots,\,\frac{\mathbf{y}_n}{\mathbf{y}_k}\Big) && (3)\\
&= \Big(\frac{\lambda\mathbf{x}_0}{\lambda\mathbf{x}_k},\,\dots,\,\frac{\lambda\mathbf{x}_{k-1}}{\lambda\mathbf{x}_k},\,\frac{\lambda\mathbf{x}_{k+1}}{\lambda\mathbf{x}_k},\,\dots,\,\frac{\lambda\mathbf{x}_n}{\lambda\mathbf{x}_k}\Big) && (4)\\
&= \Big(\frac{\mathbf{x}_0}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_{k-1}}{\mathbf{x}_k},\,\frac{\mathbf{x}_{k+1}}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_n}{\mathbf{x}_k}\Big) && (5)\\
&= \varphi_k\big([\mathbf{x}]\big) && (6)
\end{aligned}$$

So it is well-defined. It is also continuous. This is one of the characteristics of the quotient map. Given a topological space $(Y,\, \tau_Y)$ and a function $f : X/R \to Y$, f is continuous if and only if $f \circ q : X \to Y$ is continuous where $q : X \to X/R$ is the canonical quotient map. The composition $\varphi_k \circ q$ is a rational function, which is continuous, so φ_k is continuous. The inverse function is given by:

$$\varphi_k^{-1}(\mathbf{x}) = \big[(\mathbf{x}_0,\, \dots,\, \mathbf{x}_{k-1},\, 1,\, \mathbf{x}_k,\, \dots,\, \mathbf{x}_{n-1})\big] \tag{7}$$

which is continuous since the function $f : \mathbb{R}^n \to \mathbb{R}^{n+1} \setminus \{\, \mathbf{0}\, \}$ defined by:

$$f(\mathbf{x}) = (\mathbf{x}_0,\, \dots,\, \mathbf{x}_{k-1},\, 1,\, \mathbf{x}_k,\, \dots,\, \mathbf{x}_{n-1}) \tag{8}$$

is continuous, so φ_k^{-1} is the composition of continuous functions. Since the sets $\mathcal{U}_k$ cover $\mathbb{R}^{n+1} \setminus \{\, \mathbf{0}\, \}$, the sets $\tilde{\mathcal{U}}_k$ also cover $\mathbb{RP}^n$. Because of this $\mathbb{RP}^n$ is locally Euclidean. It is also second countable since it can be covered with finitely many open sets each of which is homeomorphic to an open subset of $\mathbb{R}^n$, which is hence second countable. Since $\mathbb{RP}^n$ is the finite union of second countable open subspaces, it is second countable itself. It is also Hausdorff. Given $[\mathbf{x}] \neq [\mathbf{y}]$ we have that $\mathbf{y}$ is not of the form $\lambda\mathbf{x}$ for any real number, meaning $\mathbf{x}$ and $\mathbf{y}$ lie on different lines through the origin. Let θ be defined by:

$$\theta = \frac{1}{4}\arccos\Big(\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||_2\, ||\mathbf{y}||_2}\Big) \tag{9}$$

θ is one-fourth the angle made between the lines through the origin spanned by $\mathbf{x}$ and $\mathbf{y}$. Let $\mathcal{U}$ and $\mathcal{V}$ be defined by:

$$\mathcal{U} = \big\{\, \mathbf{z} \in \mathbb{R}^{n+1} \setminus \{\, \mathbf{0}\, \} \mid \measuredangle(\mathbf{x},\, \mathbf{z}) < \theta\, \big\} \tag{10}$$
$$\mathcal{V} = \big\{\, \mathbf{z} \in \mathbb{R}^{n+1} \setminus \{\, \mathbf{0}\, \} \mid \measuredangle(\mathbf{y},\, \mathbf{z}) < \theta\, \big\} \tag{11}$$

Where $\measuredangle(\mathbf{p},\, \mathbf{q})$ is the angle between the non-zero vectors $\mathbf{p}$ and $\mathbf{q}$. These sets are open cones in $\mathbb{R}^{n+1} \setminus \{\, \mathbf{0}\, \}$ (Fig. 1) which are also saturated with respect to q, and by the choice of θ they are disjoint. But then $\tilde{\mathcal{U}} = q[\mathcal{U}]$ and $\tilde{\mathcal{V}} = q[\mathcal{V}]$ are disjoint open subsets of $\mathbb{RP}^n$ such that $[\mathbf{x}] \in \tilde{\mathcal{U}}$ and $[\mathbf{y}] \in \tilde{\mathcal{V}}$. Hence $\mathbb{RP}^n$ is Hausdorff. The real projective space is therefore a topological manifold. ■

The elements of $\mathbb{RP}^n$ are equivalence classes of $\mathbb{R}^{n+1} \setminus \{\, \mathbf{0}\, \}$. A *point* in $\mathbb{RP}^n$ is a *line* in $\mathbb{R}^{n+1}$ through the origin. It is not immediately clear that $\mathbb{RP}^n$ can be embedded as a subspace of $\mathbb{R}^N$ for some $N \in \mathbb{N}$. It indeed can, in fact $\mathbb{RP}^n$ can be embedded into $\mathbb{R}^{2n}$ for all $n > 0$, but this is by no means obvious. The case $n = 1$ is slightly obvious if you really think about what $\mathbb{RP}^1$ is (it's just a circle $\mathbb{S}^1$). The case $\mathbb{RP}^2$ is less obvious ($\mathbb{RP}^2$ is **not** a sphere). We can not embed the real projective plane into $\mathbb{R}^3$, unlike the sphere. If we try we'll end up with a surface that must intersect itself. This is shown in Fig. 2. This representation is known as the *cross cap*. We can do better than this. The cross cap has a crease in it, and this can be removed. David Hilbert, one of the pioneering mathematicians of the early 20$^{\text{th}}$ century, thought it impossible to draw the real projective plane in $\mathbb{R}^3$ in such a way that it has no crease. He asked his

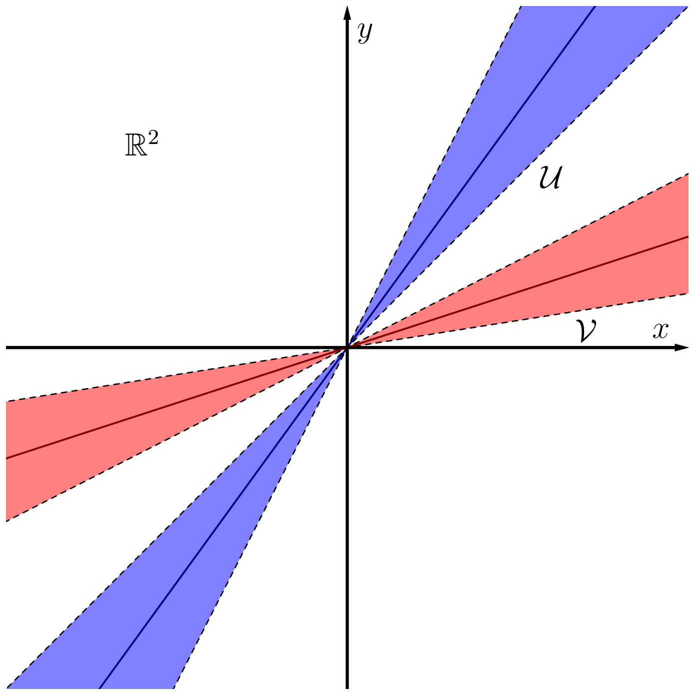

Figure 1: $\mathbb{RP}^n$ is Hausdorff

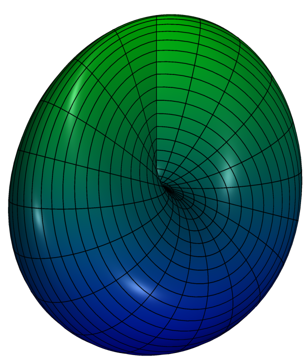
Figure 2: The Real Projective Plane

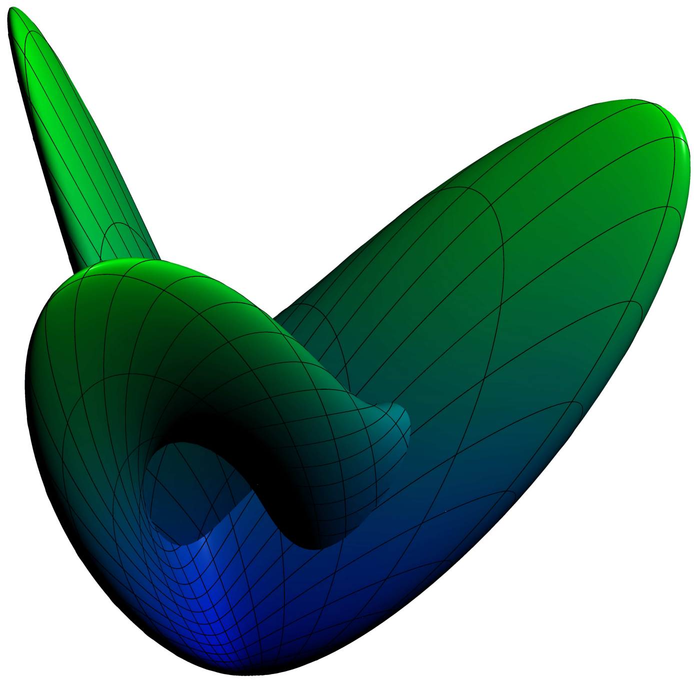

Figure 3: The Boy Surface

student Werney Boy to try and prove this. Instead Boy discovered a method of drawing the real projective plane in $\mathbb{R}^3$ that has no crease (it is still self intersecting). This is called the *Boy surface*. It is shown in Fig. 3. Bryant and Kusner discovered a way to do this using somewhat simpler functions involving complex variables. The Bryant-Kusner parameteriation is shown in Fig. 4.

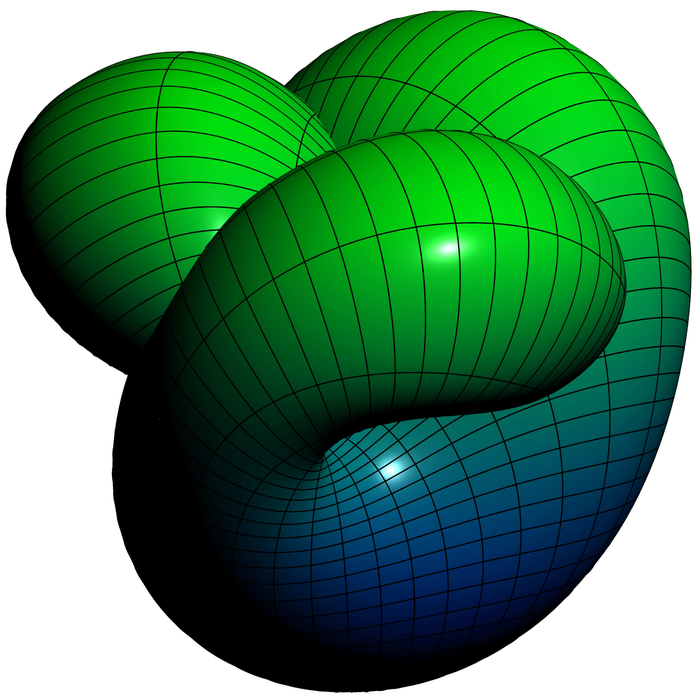

Figure 4: The Bryant-Kusner Parameterization of the Boy Surface