Point-Set Topology: Lecture 30

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1 The Topology of Manifolds

In these notes we will prove some facts about the basic topology of manifolds. For convenience, these results are summarized below.

Topological manifolds are:

- Locally compact.
- Locally metrizable.
- Regular
- Metrizable.
- Paracompact.
- Lindelöf.
- Have a countable basis of precompact coordinate balls.
- σ compact.
- Compactly exhaustible.
- Locally connected.
- Locally path-connected.
- Connected if and only if path connected.

Our goal is to prove all of these claims. But we won't add a condition if it's not needed. For example, locally Euclidean implies locally compact, there is no need to add the Hausdorff and second countability requirements.

Theorem 1.1. *If (X,τ) is a topological space, then it is locally Euclidean if and only if for all $x\in X$ there is a $\mathcal{U}\in\tau$ such that $x\in\mathcal{U}$ and $(\mathcal{U},\tau_{\mathcal{U}})$ is homeomorphic to $B_1^{\mathbb{R}^n}(\mathbf{0})$ for some $n\in\mathbb{N}$.*

Proof. If (X, τ) is locally Euclidean, then for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$, and there is an $n \in \mathbb{N}$ and a continuous injective open mapping $f : \mathcal{U} \to \mathbb{R}^n$. But since f is an open mapping, $f[\mathcal{U}] \subseteq \mathbb{R}^n$ is open. But $f(x) \in f[\mathcal{U}]$ so there is an $\varepsilon > 0$ such that $||\mathbf{y} - f(x)||_2 < \varepsilon$ implies $\mathbf{y} \in f[\mathcal{U}]$. Let $\tilde{\mathcal{V}}$ be the ε centered at $f(x)$ and $\mathcal{V} = f^{-1}[\tilde{\mathcal{V}}]$. Then $\mathcal{V} \subseteq \mathcal{U}$ is open since f is continuous. Moreover $f|_{\mathcal{V}} : \mathcal{V} \to \tilde{\mathcal{V}}$ is a continuous bijective open mapping, which is therefore a homeomorphism. But $\tilde{\mathcal{V}}$ is an open ball in $\mathbb{R}^n$ of non-zero radius, and such a ball is homeomorphic to $B_1^{\mathbb{R}^n}(\mathbf{0})$ via:

$$g(\mathbf{x}) = \frac{1}{\varepsilon}(\mathbf{x} - \mathbf{x}_0) \tag{1}$$

where $\varepsilon > 0$ is the radius and $\mathbf{x}_0$ is the center. Since homeomorphic is a transitive notion, $\mathcal{V}$ is homeomorphic to $B_1^{\mathbb{R}^n}(\mathbf{0})$ and $x \in \mathcal{V}$. In the other direction, if $x \in X$ implies there is an open set $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U}, \tau_{\mathcal{U}})$ is homeomorphic to the open ball in $\mathbb{R}^n$, then there is a homeomorphism $f : \mathcal{U} \to B_1^{\mathbb{R}^n}(\mathbf{0})$. But then, in particular, $f : \mathcal{U} \to \mathbb{R}^n$ is an injective continuous open mapping. Hence, (X, τ) is locally Euclidean. □

Theorem 1.2. *If (X, τ) is a topological space, then it is locally Euclidean if and only if for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U}, \tau_{\mathcal{U}})$ is homeomorphic to $\mathbb{R}^n$.*

Proof. By the previous theorem (X, τ) is locally Euclidean if and only if for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U}, \tau_{\mathcal{U}})$ is homeomorphic to $B_1^{\mathbb{R}^n}(\mathbf{0})$. But the open unit ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ via:

$$f(\mathbf{x}) = \frac{\mathbf{x}}{1 - ||\mathbf{x}||_2} \tag{2}$$

and hence (X, τ) is locally Euclidean if and only if for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U}, \tau_{\mathcal{U}})$ is homeomorphic to $\mathbb{R}^n$. □

Definition 1.1 (**Coordinate Ball**) A coordinate ball in a topological space (X, τ) is an open set $\mathcal{U} \in \tau$ that is homeomorphic to $\mathbb{R}^n$ (or, equivalently, homeomorphic to the unit ball $B_1^{\mathbb{R}^n}(\mathbf{0})$). ■

Theorem 1.3. *If (X, τ) is locally Euclidean, then there is a basis $\mathcal{B}$ of τ consisting only of coordinate balls.*

Proof. Let $\mathcal{B}$ be the set of all coordinate balls in (X, τ). By the previous theorems this set is an open cover of X. It is also a basis. For given $\mathcal{U} \in \tau$ and $x \in \mathcal{U}$, since $x \in X$ there is a coordinate ball $\mathcal{V} \in \tau$ such that $x \in \mathcal{V}$. But then there is a homeomorphism $f : \mathcal{V} \to \mathbb{R}^n$ for some $n \in \mathbb{N}$. Since $\mathcal{U}$ is open, $\mathcal{U} \cap \mathcal{V} \subseteq \mathcal{V}$ is open. But then, since f is a homeomorphism, $f[\mathcal{U} \cap \mathcal{V}]$ is an open subset of $\mathbb{R}^n$. And since $x \in \mathcal{U} \cap \mathcal{V}$ we have $f(x) \in f[\mathcal{U} \cap \mathcal{V}]$. So there is an $\varepsilon > 0$ such that the open ball of radius ε centered at $f(x)$ sits inside of $f[\mathcal{U} \cap \mathcal{V}]$. Label this open ball as $\tilde{\mathcal{W}}$. But then $\mathcal{W} = f^{-1}[\tilde{\mathcal{W}}]$ is a coordinate ball in (X, τ), so $\mathcal{W} \in \mathcal{B}$, and it is such that $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{V}$. Therefore $\mathcal{B}$ is a basis. □

Theorem 1.4. *If (X, τ) is locally Euclidean and Lindelöf, then there is a countable basis $\mathcal{B}$ of coordinate balls for τ. In particular, (X, τ) is second countable.*

Proof. Let $\mathcal{B}$ be the set of all coordinate ball in (X, τ). By the previous theorem this is a basis, and hence an open cover of X. But (X, τ) is Lindelöf so there is a countable subcover $\Delta \subseteq \mathcal{B}$. Then for all $\mathcal{U} \in \Delta$ we have that $\mathcal{U}$ is a coordinate ball, so homeomorphic to $\mathbb{R}^n$. But $\mathbb{R}^n$ has a countable basis consisting of open balls, meaning $\mathcal{U} \in \Delta$ has a countable basis of coordinate balls. Let $\mathcal{B}_{\mathcal{U}}$ be such a basis for $\mathcal{U}$ and define:

$$\mathcal{B} = \bigcup_{\mathcal{U} \in \Delta} B_{\mathcal{U}} \tag{3}$$

This is the countable union (since Δ is countable) of countable sets (since each $\mathcal{B}_{\mathcal{U}}$ is countable) and hence $\mathcal{B}$ is countable. It is also a basis since each $\mathcal{B}_{\mathcal{U}}$ is a basis for $\mathcal{U}$ and the set of all $\mathcal{U} \in \Delta$ cover X. Hence $\mathcal{B}$ is a countable basis of coordinate balls. Since it is a countable basis, (X, τ) is second countable. □

Definition 1.2 (**Precompact**) A precompact subset of a topological space (X, τ) is a subset $A \subseteq X$ such that $\mathrm{Cl}_\tau(A)$ is compact. ■

Theorem 1.5. *If (X, τ) is locally Euclidean and Hausdorff, then for all $x \in X$ there is a precompact coordinate ball $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$.*

Proof. Since (X, τ) is locally Euclidean, then is a coordinate ball $\mathcal{V} \in \tau$ such that $x \in \mathcal{V}$. Let $f : \mathcal{V} \to \mathbb{R}^n$ be a homeomorphism such that $f(x) = \mathbf{0}$ (we can always do this by translating). Let $\tilde{\mathcal{U}} = B_1^{\mathbb{R}^n}(\mathbf{0})$ and $\mathcal{U} = f^{-1}[\tilde{\mathcal{U}}]$. Then, since f is a homeomorphism it is continuous, so $\mathcal{U} \subseteq \mathcal{V}$ is an open subset. But $\mathrm{Cl}_{\tau_{\mathbb{R}^n}}(\tilde{\mathcal{V}})$ is compact, since it is the closed unit ball in $\mathbb{R}^n$. So $f^{-1}[\mathrm{Cl}_{\tau_{\mathbb{R}^n}}(\tilde{\mathcal{V}})]$ is a compact subset of $\mathcal{U}$ since f is a homeomorphisn, and since (X, τ) is Hausdorff, this set is a closed subset as well. But then:

$$f^{-1}[\mathrm{Cl}_{\tau_{\mathbb{R}^n}}(\tilde{\mathcal{V}})] = \mathrm{Cl}_{\tau_{\mathcal{U}}}(\mathcal{V}) \tag{4}$$

$$= \mathrm{Cl}_\tau(\mathcal{V}) \tag{5}$$

and hence $\mathcal{V} \subseteq X$ is a precompact coordinate ball that contains x. □

Note in the previous theorem we considered the closure with respect to three different topologies. First, the standard Euclidean topology in $\mathbb{R}^n$. Next, the subspace topology of the open set $\mathcal{U}$. Lastly, the topology on X. Since X is Hausdorff, $\mathrm{Cl}_{\tau_{\mathcal{U}}}(\mathcal{V})$ being compact implies it is closed. Because of this the closure with respect to the subspace topology $\tau_{\mathcal{U}}$ is the same as the closure with respect to the ambient topology τ, meaning $\mathcal{V}$ is precompact in (X, τ). The Hausdorff property is essential. Without it the closure with respect to the subpace $\mathcal{U}$ and the entire space X may be very different, and $\mathrm{Cl}_\tau(X)$ might fail to be compact.

Theorem 1.6. *If (X, τ) is locally Euclidean and Hausdorff, then there is a basis $\mathcal{B}$ of precompact coordinate balls.*

Proof. Let $\mathcal{B}$ be the set of all precompact coordinate balls in (X, τ). By the previous theorem $\mathcal{B}$ is an open cover. It is also a basis. Given $\mathcal{U} \in \tau$ and $x \in \mathcal{U}$ we can find a precompact coordinate ball $\mathcal{V} \in \mathcal{B}$ such that $x \in \mathcal{V}$. That is, there is a homeomorphism $f : \mathcal{V} \to \mathbb{R}^n$. Since $x \in \mathcal{U} \cap \mathcal{V}$, and since f is a homeomorphism, $f[\mathcal{U} \cap \mathcal{V}]$ is an open subset of $\mathbb{R}^n$ and $f(x) \in f[\mathcal{U} \cap \mathcal{V}]$. So there is an $\varepsilon > 0$ such that the ε ball about $f(x)$ is a subset of $f[\mathcal{U} \cap \mathcal{V}]$. Let $\tilde{\mathcal{W}}$ be the ε ball centered about $f(x)$. Then $\mathcal{W} = f^{-1}[\tilde{\mathcal{W}}]$ is a coordinate ball that sits inside of $\mathcal{U} \cap \mathcal{V}$, and since $\mathcal{V}$ is precompact, $\mathcal{W}$ is precompact as well since $\mathrm{Cl}_\tau(\mathcal{W}) \subseteq \mathrm{Cl}_\tau(\mathcal{V})$. Hence $\mathcal{W}$ is a precompact coordinate ball such that $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{U}$. Therefore $\mathcal{B}$ is a basis for τ. □

Theorem 1.7. *If (X, τ) is a topological manifold, then it is Lindelöf.*

Proof. By definition topological manifolds are second countable. But second countable spaces are Lindelöf, and hence (X, τ) is Lindelöf. □

Theorem 1.8. *If (X, τ) is a topological manifold, then there exists a countable basis $\mathcal{B}$ of τ consisting of precompact coordinate balls.*

Proof. The proof is now a combination of a few previous ideas. We just proved there is a basis $\mathcal{B}$ of precompact coordinate balls. Since topological manifolds are Lindelöf, there is a countable subcover $\Delta \subseteq \mathcal{B}$. So countably many precompact coordinate balls cover X, each of which has a countable basis of precompact coordinate balls. The union of all of these bases for every element of Δ is a basis for τ, and since it is the countable union of countable sets, this union is countable itself. So τ has a countable basis of precompact coordinate balls. □

Theorem 1.9. *If (X, τ) is locally Euclidean, then it is locally metrizable.*

Proof. Every point $x \in X$ has an open set $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{U}$ is homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. But $\mathbb{R}^n$ is metrizable, the topology being induced by the Euclidean metric, so $\mathcal{U}$ is a metrizable subspace of (X, τ) that contains x. Hence, (X, τ) is locally metrizable. □

Theorem 1.10. *If (X, τ) is locally Euclidean, then it is locally compact.*

Proof. Let $x \in X$. Since (X, τ) is locally Euclidean, there is a coordinate ball $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{U}$ is homeomorphic to $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathcal{U}$ be a homeomorphism. Then in particular $f : \mathbb{R}^n \to X$ is an injective continuous open mapping. Let $\tilde{\mathcal{V}}$ be the open ball of radius 1 centered about $f(x)$ and let $\tilde{K}$ be the closed ball of radius 1 centered at $f(x)$. Then $\tilde{\mathcal{V}} \subseteq \tilde{K}$. By the Heine-Borel theorem $\tilde{K}$ is compact. But then, since f is continuous, $K = f[\tilde{K}] \subseteq X$ is compact. Let $\mathcal{V} = f[\tilde{\mathcal{V}}]$. Since $\tilde{\mathcal{V}}$ is open and f is an open mapping, $\mathcal{V} \subseteq \mathcal{U}$ is open. But $x \in \mathcal{V}$ and $\mathcal{V} \subseteq K$. Hence, (X, τ) is locally compact. □

Theorem 1.11. *If (X, τ) is locally Euclidean and Hausdorff, then it is regular.*

Proof. Since (X, τ) is locally Euclidean, it is locally compact. But locally compact Hausdorff spaces are regular, and hence (X, τ) is regular. □

Theorem 1.12. *If $(X,\, \tau)$ is a topological manifold, then it is metrizable.*

Proof. Since $(X,\, \tau)$ is a topological manifold, it is Hausdorff and second countable by definition. But topological manifolds are also locally Euclidean and locally Euclidean Hausdorff spaces are regular. Hence $(X,\, \tau)$ is regular, Hausdorff, and second countable, so by Urysohn's metrization theorem it is metrizable. □

Theorem 1.13. *If $(X,\, \tau)$ is a topological manifold, then it is paracompact.*

Proof. Since $(X,\, \tau)$ is a topological manifold, it is metrizable. But by Stone's theorem metrizable spaces are paracompact. Hence, $(X,\, \tau)$ is paracompact. □

Theorem 1.14. *If $(X,\, \tau)$ is locally Euclidean, Hausdorff, and paracompact, then it is metrizable.*

Proof. Since locally Euclidean space are locally metrizable, $(X,\, \tau)$ is a locally metrizable Hausdorff space that is paracompact. By Smirnov's theorem, $(X,\, \tau)$ is metrizable. □

Theorem 1.15. *If $(X,\, \tau)$ is a topological manifold, then it is σ compact.*

Proof. Since $(X,\, \tau)$ is a topological manifold, there is a countable basis $\mathcal{B}$ of precompact coordinate balls. Let $\mathcal{U} : \mathbb{N} \to \mathcal{B}$ be a surjection. Then, since $\mathcal{B}$ is a basis, we have $X = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. But also, since all elements of $\mathcal{B}$ are precompact, for all $n \in \mathbb{N}$ it is true that $\mathrm{Cl}_\tau(\mathcal{U}_n)$ is compact. Let $K_n = \mathrm{Cl}_\tau(\mathcal{U}_n)$. Then each K_n is compact and $\bigcup_{n\in\mathbb{N}} K_n = X$. Hence $(X,\, \tau)$ is σ compact. □

Theorem 1.16. *If $(X,\, \tau)$ is a topological manifold, it is compactly exhaustible.*

Proof. Since $(X,\, \tau)$ is a topological manifold, it is σ compact. But topological manifolds are also Hausdorff and locally compact. But locally compact Hausdorff spaces that are σ compact are compactly exhaustible. Hence, $(X,\, \tau)$ is compactly exhaustible. □

We've already stated this theorem, but let's prove it again.

Theorem 1.17. *If $(X,\, \tau)$ is a topological manifold, then it is paracompact.*

Proof. Since $(X,\, \tau)$ is a topological manifold, it is compactly exhaustible. But compactly exhaustible Hausdorff spaces are paracompact. Hence, $(X,\, \tau)$ is paracompact. □

The connectedness theorems come from the locally Euclidean property.

Theorem 1.18. *If $(X,\, \tau)$ is locally Euclidean, then it is locally connected.*

Proof. Since $(X,\, \tau)$ is locally Euclidean it has a basis of coordinate balls. But coordinate balls are connected, so $(X,\, \tau)$ is locally connected. □

Theorem 1.19. *If $(X,\, \tau)$ is locally Euclidean, then it is locally path connected.*

Proof. Since (X, τ) is locally Euclidean it has a basis of coordinate balls. But coordinate balls are path connected, so (X, τ) is locally path connected. □

Theorem 1.20. *If (X, τ) is locally Euclidean, then it is connected if and only if it is path connected.*

Proof. Path connected always implies connected. If (X, τ) is locally Euclidean and connected, then since locally Euclidean spaces are locally path connected, (X, τ) must be path connected. □