Completing a Metric Space

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Summer 2022

Definition 1 (**Dense Subspace in a Metric Space**) A dense subspace in a metric space $(X,\,d)$ is a metric subspace $(A,\,d_A)$, $A\subseteq X$, such that for every $x\in X$ there is a sequence $a:\mathbb{N}\to A$ such that $a_n\to x$. That is, a sequence in A that converges to x in $(X,\,d)$. ■

The motivating examples are the rational and irrational numbers in the real line. Every real number can be approximated arbitrarily well by a rational number, and every real number can also be approximated by an irrational number.

Definition 2 (**Completion of a Metric Space**) A completion of a metric space $(X,\,d)$ is a complete metric space $(\tilde{X},\,\tilde{d})$ such that there is an isometry $f:X\to\tilde{X}$ where $f[X]\subseteq\tilde{X}$ is a dense subspace. ■

We will prove in these notes that every metric space $(X,\,d)$ has a completion, and that this completion is essentially unique.

Theorem 1. *If $(X,\,d)$ is a metric space, and if $a,b:\mathbb{N}\to X$ are Cauchy sequences, then the sequence $r:\mathbb{N}\to\mathbb{R}$ defined by $r_n=d(a_n,\,b_n)$ is bounded.*

Proof. Let $\varepsilon=1$. Since a and b are Cauchy, there are $N_0,N_1\in\mathbb{N}$ such that $m,n>N_0$ implies $d(a_m,\,a_n)<\varepsilon$ and $m,n>N_1$ implies $d(b_m,\,b_n)<\varepsilon$. Let $N=\max(N_0,\,N_1)$ and let $M=\max\big(d(a_0,\,b_0),\,\dots,\,d(a_{N+1},\,b_{N+1})\big)+2$. M is a bound for r. For given any $n\in\mathbb{N}$, if $n\leq N$ we have:

$$r_n=d(a_n,\,b_n)\leq\max\big(d(a_0,\,b_0),\,\dots,\,d(a_{N+1},\,b_{N+1})\big)<M \tag{1}$$

by definition of M. If $n>N$ we get:

$$\begin{aligned}
r_n&=d(a_n,\,b_n) && (2)\\
&\leq d(a_n,\,a_{N+1})+d(a_{N+1},\,b_{N+1})+d(b_{N+1},\,b_n) && (3)\\
&<\varepsilon+\max\big(d(a_0,\,b_0),\,\dots,\,d(a_{N+1},\,b_{N+1})\big)+\varepsilon && (4)\\
&=2+\max\big(d(a_0,\,b_0),\,\dots,\,d(a_{N+1},\,b_{N+1})\big) && (5)\\
&=M && (6)
\end{aligned}$$

So r_n is bounded between 0 and $M+2$, and so is bounded. □

Theorem 2 (**The Trapezoid Inequality**). *If $(X,\,d)$ is a metric space, if $a,b,c,d\in X$, then $|d(a,\,c)-d(b,\,d)|\leq d(a,\,b)+d(c,\,d)$.*

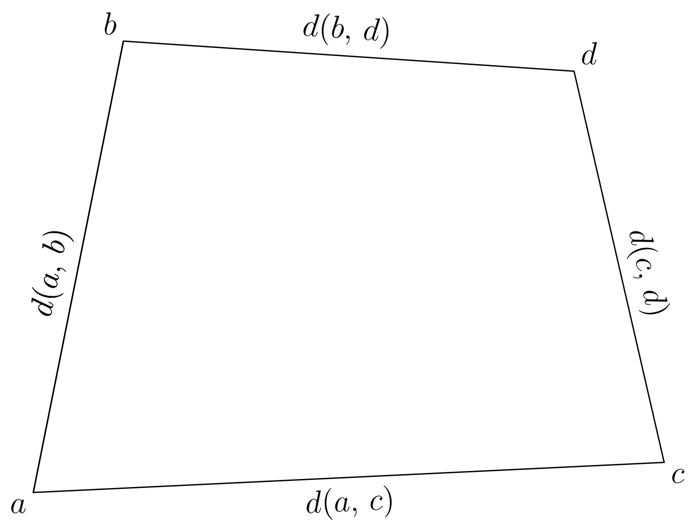

Figure 1: The Trapezoid Inequality

Proof. There are two cases, $d(a,\, c) \geq d(b,\, d)$ and $d(a,\, c) \leq d(b,\, d)$. Suppose $d(a,\, c) \geq d(b,\, d)$. The argument is symmetric in the other case. Then:

$$|d(a,\, c) - d(b,\, d)| = d(a,\, c) - d(b,\, d) \tag{7}$$

By the triangle inequality, $d(b,\, c) \leq d(b,\, d) + d(c,\, d)$, and therefore:

$$d(b,\, c) - d(b,\, d) \leq d(c,\, d) \tag{8}$$

Using this we then have:

$$\begin{aligned} |d(a,\, c) - d(b,\, d)| &= d(a,\, c) - d(b,\, d) && (9) \\ &\leq d(a,\, b) + d(b,\, c) - d(b,\, d) && (10) \\ &\leq d(a,\, b) + d(c,\, d) && (11) \end{aligned}$$

Completing the proof. □

The trapezoid inequality gets its name from Fig. 1.

Theorem 3. *If $(X,\, d)$ is a metric space, if $a, b : \mathbb{N} \to X$ are Cauchy sequences, and if $r : \mathbb{N} \to \mathbb{R}$ is defined by $r_n = d(a_n,\, b_n)$, then r is a convergent sequence in $\mathbb{R}$.*

Proof. Let $\varepsilon > 0$. Since a and b are Cauchy, there is an $N_0, N_1 \in \mathbb{N}$ such that $m, n > N_0$ implies $d(a_m,\, a_n) < \varepsilon/4$ and $m, n > N_1$ implies $d(b_m,\, b_m) < \varepsilon/4$.

Let $N = \max(N_0, N_1) + 1$. Then by the trapezoid inequality, $m, n > N$ implies:

$$\begin{aligned}
|r_m - r_n| &= |r_m - r_N + r_N - r_n| && (12)\\
&\le |r_m - r_N| + |r_N - r_n| && (13)\\
&= |d(a_m, b_m) - d(a_N, b_N)| + |d(a_N, b_N) - d(a_n, b_n)| && (14)\\
&\le d(a_N, a_m) + d(b_N, b_m) + d(a_N, a_n) + d(b_N, b_n) && (15)\\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} && (16)\\
&= \varepsilon && (17)
\end{aligned}$$

so r is a Cauchy sequence. But $(\mathbb{R}, |\cdot|)$ is complete, so Cauchy sequences converge. Hence, r is a convergent sequence. □

Theorem 4. *If (X, d) is a metric space, if $\mathcal{A}$ is the set of all Cauchy sequences $a : \mathbb{N} \to X$, and if R is the relation defined on $\mathcal{A}$ by aRb if and only if $d(a_n, b_n) \to 0$, then R is an equivalence relation on $\mathcal{A}$.*

Proof. R is reflexive since $d(a_n, a_n) = 0$, so aRa. R is symmetric since aRb implies $d(a_n, b_n) \to 0$, but $d(a_n, b_n) = d(b_n, a_n)$, so $d(b_n, a_n) \to 0$, and hence bRa. Lastly, it is transitive. If aRb and bRc, then $d(a_n, c_n) \le d(a_n, b_n) + d(b_n, c_n)$, and both of these latter two expressions tend to zero since aRb and bRc, so $d(a_n, c_n) \to 0$. That is, aRc. So R is reflexive, symmetric, and transitive, and is therefore an equivalence relation. □

Theorem 5. *If (X, d) is a metric space, if $\mathcal{A}$ is the set of all Cauchy sequences $a : \mathbb{N} \to X$, if R is equivalence relation aRb if and only if $d(a_n, b_n) \to 0$, if $\tilde{X} = \mathcal{A}/R$, and if $\tilde{d}$ is defined by the formula:*

$$\tilde{d}\big([a], [b]\big) = \lim_{n\to\infty} d(a_n, b_n) \tag{18}$$

then $\tilde{d}$ is a well-defined function $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$.

Proof. Let $a, b, c, d \in \mathcal{A}$ be Cauchy sequences with aRc and bRd. Then $d(a_n, c_n) \to 0$ and $d(b_n, d_n) \to 0$. But then:

$$\begin{aligned}
\tilde{d}\big([c], [d]\big) &= \lim_{n\to\infty} d(c_n, d_n) && (19)\\
&\le \lim_{n\to\infty} \big(d(a_n, c_n) + d(a_n, d_n)\big) && (20)\\
&\le \lim_{n\to\infty} \big(d(a_n, c_n) + d(a_n, b_n) + d(b_n, d_n)\big) && (21)\\
&= \lim_{n\to\infty} d(a_n, c_n) + \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} d(b_n, d_n) && (22)\\
&= 0 + \lim_{n\to\infty} d(a_n, b_n) + 0 && (23)\\
&= \lim_{n\to\infty} d(a_n, b_n) && (24)\\
&= \tilde{d}\big([a], [b]\big) && (25)
\end{aligned}$$

so $\tilde{d}$ is well-defined. □

Theorem 6. *If* $(X,\, d)$ *is a metric space, if* $\mathcal{A}$ *is the set of all Cauchy sequences* $a:\mathbb{N}\rightarrow X$*, and if* R *is the equivalence relation* aRb *if and only if* $d(a_n,\, b_n)\rightarrow 0$*, if* $\tilde{X}=\mathcal{A}/R$*, and if* $\tilde{d}$ *is the function:*

$$\tilde{d}\big([a],\,[b]\big)=\lim_{n\rightarrow\infty}d(a_n,\,b_n) \tag{26}$$

then $(\tilde{X},\,\tilde{d})$ *is a complete metric space.*

Proof. $\tilde{d}$ is indeed a metric. Since $(X,\, d)$ is a metric space, $\tilde{d}$ is non-negative since d is non-negative. Also:

$$\begin{aligned}&\tilde{d}\big([a],\,[b]\big)=0 &(27)\\ \Leftrightarrow\, &d(a_n,\,b_n)\rightarrow 0 &(28)\\ \Leftrightarrow\, &aRb &(29)\\ \Leftrightarrow\, &[a]=[b] &(30)\end{aligned}$$

so $\tilde{d}$ is positive-definite. It is symmetric since:

$$\begin{aligned}\tilde{d}\big([a],\,[b]\big)&=\lim_{n\rightarrow\infty}d(a_n,\,b_n) &(31)\\ &=\lim_{n\rightarrow\infty}d(b_n,\,a_n) &(32)\\ &=\tilde{d}\big([b],\,[a]\big) &(33)\end{aligned}$$

Lastly, it satisfies the triangle inequality. Given $[a]$, $[b]$, and $[c]$, we have:

$$\begin{aligned}\tilde{d}\big([a],\,[b]\big)&=\lim_{n\rightarrow\infty}d(a_n,\,b_n) &(34)\\ &\leq\lim_{n\rightarrow\infty}\big(d(a_n,\,c_n)+d(c_n,\,b_n)\big) &(35)\\ &=\lim_{n\rightarrow\infty}d(a_n,\,c_n)+\lim_{n\rightarrow\infty}d(c_n,\,b_n) &(36)\\ &=\tilde{d}\big([a],\,[c]\big)+\tilde{d}\big([c],\,[b]\big) &(37)\end{aligned}$$

It is also complete. Let $\mathbf{x}:\mathbb{N}\rightarrow\tilde{X}$ be a Cauchy sequence. That is, $\mathbf{x}$ is a *sequence of equivalence classes of Cauchy sequences*. For every $n\in\mathbb{N}$ there is a Cauchy sequence $x^n:\mathbb{N}\rightarrow X$ such that $\mathbf{x}_n=[x^n]$. Since x^n is a Cauchy sequence, there is an $N_n\in\mathbb{N}$ such that $k,\ell>N_n$ implies $d(x^n_k,\,x^n_\ell)<\frac{1}{n+1}$. Define $a:\mathbb{N}\rightarrow X$ by $a_n=x^n_{N_n}$. We now must show that a is a Cauchy sequence and that $\mathbf{x}_n\rightarrow[a]$. Let $\varepsilon>0$. Let M_0 be such that $N_{M_0}+1>3/\varepsilon$. Since $\mathbf{x}$ is Cauchy there is an $M_1\in\mathbb{N}$ such that $k,\ell\in\mathbb{N}$ and $k,\ell>M_1$ implies $\tilde{d}(\mathbf{x}_k,\,\mathbf{x}_\ell)<\varepsilon/3$. That is:

$$\tilde{d}(\mathbf{x}_k,\,\mathbf{x}_\ell)=\lim_{n\rightarrow\infty}d(x^k_n,\,x^\ell_n)<\frac{\varepsilon}{3} \tag{38}$$

Let $M=\max(M_0,\,M_1)+1$. Then $m,n>M$ implies:

$$\begin{aligned}d(a_m,\,a_n)&=d(x^m_{N_m},\,x^n_{N_n}) &(39)\\ &\leq d(x^m_{N_m},\,x^m_M)+d(x^m_M,\,x^n_M)+d(x^n_M,\,x^n_{N_n}) &(40)\\ &<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} &(41)\\ &=\varepsilon &(42)\end{aligned}$$

So a is a Cauchy sequence. Now we must show that $\mathbf{x}_n \to [a]$. We have:

$$\tilde{d}([a],\, \mathbf{x}_n) = \lim_{m\to\infty} d(a_m,\, x^n_m) \tag{43}$$

$$= \lim_{m\to\infty} d(x^m_{N_m},\, x^n_m) \tag{44}$$

and this converges to zero as n tends to infinity. So, $\mathbf{x}_n \to [a]$ and thus $(\tilde{X},\, \tilde{d})$ is complete. □

Theorem 7. *If $(X,\, d)$ is a metric space, if $\mathcal{A}$ is the set of all Cauchy sequences $a : \mathbb{N} \to X$, and if R is the equivalence relation aRb if and only if $d(a_n,\, b_n) \to 0$, if $\tilde{X} = \mathcal{A}/R$, and if $\tilde{d}$ is the function:*

$$\tilde{d}\big([a],\, [b]\big) = \lim_{n\to\infty} d(a_n,\, b_n) \tag{45}$$

then there is an isometry $f : X \to \tilde{X}$ into the complete metric space $(\tilde{X},\, \tilde{d})$ such that $f[X] \subseteq \tilde{X}$ is a dense subspace.

Proof. Given $x \in X$, define $g : X \to \mathcal{A}$ via $g(x) = a$ where $a : \mathbb{N} \to X$ is the sequence $a_n = x$. Since a is a constant sequence, it is a Cauchy sequence. Let $f : X \to \tilde{X}$ be defined by $f(x) = [g(x)]$. f is an isometry. For if $x, y \in X$, then:

$$\tilde{d}\big(f(x),\, f(y)\big) = \tilde{d}\big([g(x)],\, [g(y)]\big) \tag{46}$$

$$= \lim_{n\to\infty} d\big(g(x)_n,\, g(y)_n\big) \tag{47}$$

$$= \lim_{n\to\infty} d(x,\, y) \tag{48}$$

$$= d(x,\, y) \tag{49}$$

and hence, f is an isometry. Moreover, $f[X]$ is a dense subset of $\tilde{X}$. Let $[a] \in \tilde{X}$ where $a \in \mathcal{A}$ is a Cauchy sequence. Define the sequence $\mathbf{x} : \mathbb{N} \to f[X]$ via $\mathbf{x}_n = f(a_n)$. Then:

$$\lim_{n\to\infty} \tilde{d}([a],\, \mathbf{x}_n) = \lim_{n\to\infty} \lim_{m\to\infty} d(a_m,\, a_n) \tag{50}$$

$$= 0 \tag{51}$$

so $\mathbf{x}$ is a sequence in $f[X]$ that converges to $[a]$ in $(\tilde{X},\, \tilde{d})$, and hence $f[X]$ is dense. □

This is essentially the unique metric space that completes $(X,\, d)$. If $(Y,\, d_Y)$ is another complete metric space such that there exists an isometry $f : X \to Y$ such that $f[X] \subseteq Y$ is a dense subspace, then there is a global isometry between $(Y,\, d_Y)$ and $(\tilde{X},\, \tilde{d})$.