Geometric Series

Ryan Maguire

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Given a real number $r \in \mathbb{R}$ with $|r| < 1$ it can be shown that r^n tends to zero as n increases to infinity. This fact will be used freely throughout. Consider the series of partial sums of r^n:

$$S_N = \sum_{n=0}^{N} r^n \tag{1}$$

We can rewrite this in a smaller closed-form. Multiplying both sides by $1 - r$ we get:

$$\begin{aligned}
S_N(1-r) &= (1-r)\sum_{n=0}^{N} r^n && (2)\\
&= \sum_{n=0}^{N} r^n(1-r) && (3)\\
&= \sum_{n=0}^{N} \big(r^n - r^{n+1}\big) && (4)\\
&= \sum_{n=0}^{N} r^n - \sum_{n=0}^{N} r^{n+1} && (5)\\
&= 1 + \sum_{n=1}^{N} r^n - \sum_{n=0}^{N} r^{n+1} && (6)\\
&= 1 + \sum_{n=1}^{N} r^n - r^{N+1} - \sum_{n=0}^{N-1} r^{n+1} && (7)\\
&= (1 - r^{N+1}) + \sum_{n=1}^{N} r^n - \sum_{n=0}^{N-1} r^{n+1} && (8)\\
&= (1 - r^{N+1}) + \sum_{n=1}^{N} r^n - \sum_{n=1}^{N} r^n && (9)\\
&= 1 - r^{N+1} && (10)
\end{aligned}$$

Dividing both sides by $1 - r$ gives us:

$$S_N = \frac{1 - r^{N+1}}{1 - r} \tag{11}$$

Since r^{N+1} tends to zero as N increases to infinity, we get the final result:

$$\sum_{n=0}^{\infty} = \frac{1}{1 - r} \tag{12}$$

Let's examine the tail end.

$$\begin{align}
\sum_{n=N+1}^{\infty} r^n &= \sum_{n=0}^{\infty} r^n - \sum_{n=0}^{N} r^n \tag{13}\\
&= \frac{1}{1 - r} - \sum_{n=0}^{N} r^n \tag{14}\\
&= \frac{1}{1 - r} - \frac{1 - r^{N+1}}{1 - r} \tag{15}\\
&= \frac{r^{N+1}}{1 - r} \tag{16}
\end{align}$$

Since r^{N+1} gets smaller as N gets larger we see that the tail end converges to zero (as it must for the series to converge).

We can prove the partial sum formula using induction as well. The base case $N = 0$ says:

$$\sum_{n=0}^{0} r^n = r^0 = 1 = \frac{1 - r^{0+1}}{1 - r} = \frac{1 - r}{1 - r} \tag{17}$$

which is true. Suppose the formula holds for $N \in \mathbb{N}$. Then:

$$\begin{align}
\sum_{n=0}^{N+1} r^n &= r^{N+1} + \sum_{n=0}^{N} r^n \tag{18}\\
&= r^{N+1} + \frac{1 - r^{N+1}}{1 - r} \tag{19}\\
&= \frac{r^{N+1}(1 - r) + 1 - r^{N+1}}{1 - r} \tag{20}\\
&= \frac{r^{N+1} - r^{N+2} + 1 - r^{N+1}}{1 - r} \tag{21}\\
&= \frac{1 - r^{N+2}}{1 - r} \tag{22}
\end{align}$$

and so the formula holds for $N + 1$ as well.