Mathematical Induction

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Peano arithmetic is a weaker system of axioms that is sufficient to prove a lot of the elementary properties of the natural numbers. The system of set theory we have been working with, Zermelo-Fraenkel set theory with the the axiom of choice (ZFC) contains Peano arithmetic as a *subset* (via the axiom of infinity, and a few others). This system postulates the following:

1. There is a set $\mathbb{N}$ with $0 \in \mathbb{N}$.
2. There is a function $\sigma : \mathbb{N} \to \mathbb{N}$.
3. If $m, n \in \mathbb{N}$ and $\sigma(m) = \sigma(n)$, then $m = n$.
4. $\sigma(n) = 0$ is always false.
5. If $A \subseteq \mathbb{N}$ is such that $0 \in A$ and $n \in A$ implies $\sigma(n) \in A$, then $A = \mathbb{N}$.

This is constructable in ZFC, the function σ is the *plus one* function, $\sigma(n) = n + 1$. This last property is crucial. It says if $A \subseteq \mathbb{N}$ is such that $0 \in A$ and $n \in A$ implies $n+1 \in A$, then $A = \mathbb{N}$. The intuition goes like this. $0 \in A$ is true. But $0 \in A$ implies $0 + 1 \in A$, so $1 \in A$. But $1 \in A$ implies $1 + 1 \in A$, so $2 \in A$. But $2 \in A$ implies $2 + 1 \in A$, so $3 \in A$. And so on. This fact is also provable in ZFC (it is an *axiom* in Peano arithmetic). Containment $\in$ defines an order on the integers. Remember, in ZFC, that $0 = \emptyset$, $1 = \{\, 0 \,\}$, $2 = \{\, 0,\, 1 \,\}$, and so on. We write $m < n$ if and only if $m \in n$. This defines a *well-order*. Every non-empty subset of $\mathbb{N}$ has a *smallest* element. This well-ordering property gives us property 5 in Peano arithmetic.

Let P be a predicate on the natural numbers. That is, for every $n \in \mathbb{N}$, $P(n)$ is a sentence which we may say is true or false. Suppose $P(0)$ is true, and the truth of $P(n)$ implies the truth of $P(n + 1)$. Then $P(n)$ is true for all $n \in \mathbb{N}$. Why? Well, $P(1)$ is true since $P(0)$ is true, and $P(0)$ being true implies $P(0+1)$ is true. Then $P(2)$ is true since $P(1)$ is true and $P(1)$ being true implies $P(2)$ is true, and so on. We can prove this rigorously.

Theorem 1 (**Principle of Mathematical Induction**)**.** *If P is a predicate on the natural numbers such that $P(0)$ is true and for all $n \in \mathbb{N}$ the truth of $P(n)$ implies the truth of $P(n + 1)$, then $P(n)$ is true for all $n \in \mathbb{N}$.*

Proof. Suppose not. Then there is some $m \in \mathbb{N}$ such that $P(m)$ is false. Define A via:

$$A = \{ k \in \mathbb{N} \mid P(k) \text{ is false.} \} \tag{1}$$

Since $m \in \mathbb{N}$, A is non-empty. So there is a least element $N \in A$. But $N \neq 0$ since $P(0)$ is true by hypothesis. Since $N \neq 0$ and $N \in \mathbb{N}$, we can write $N = n + 1$ for some $n \in \mathbb{N}$. But then $n < N$, and since N is the least element of A, $P(n)$ must be true. But the truth of $P(n)$ implies the truth of $P(n+1)$, so $P(n+1)$ is true. But $n+1 = N$, so $P(N)$ is true, a contradiction. Hence, $P(n)$ is true for all $n \in \mathbb{N}$. □

Let's use this.

Example 1 Consider the partial sums S_N for $N \in \mathbb{N}$ defined by:

$$S_N = \sum_{n=0}^{N} n \tag{2}$$

We can provide a closed-form formula for this via the principle of induction. We want to prove that:

$$S_N = \frac{N(N+1)}{2} \tag{3}$$

That is, $P(N)$ is the predicate on the natural numbers that $S_N = N(N+1)/2$. We prove $P(N)$ is true for all $N \in \mathbb{N}$ via induction. The case $N = 0$ says $0 = 0(0+1)/2$, which is true. Suppose the statement is true for some $N \in \mathbb{N}$. We must prove this implies the statement is true for $N + 1$. We compute:

$$S_{N+1} = \sum_{n=0}^{N+1} n \tag{4}$$

$$= N + 1 + \sum_{n=0}^{N} n \tag{5}$$

$$= N + 1 + S_N \tag{6}$$

But $P(N)$ is true, by hypothesis, so $S_N = N(N+1)/2$. We get:

$$S_{N+1} = N + 1 + S_N \tag{7}$$

$$= N + 1 + \frac{N(N+1)}{2} \tag{8}$$

$$= \frac{2N + 2 + N^2 + N}{2} \tag{9}$$

$$= \frac{N^2 + 3N + 2}{2} \tag{10}$$

$$= \frac{(N+1)(N+2)}{2} \tag{11}$$

and therefore $P(N+1)$ is true. Since the formula is true for $N = 0$ and $P(N)$ implies $P(N+1)$, we see that by the principle of induction $P(N)$ is true for all $N \in \mathbb{N}$. ■