Point-Set Topology: Homework 1

Summer 2023

Problem 1 (Hilbert Systems)

The Hilbert System is a collection of axioms for how propositional logic should behave. It claims the following four statements are true and do not need proof. Let P, Q, and R be propositions (statements that are true or false). Then the following are true:

$$P \Rightarrow P \tag{1}$$
$$P \Rightarrow (Q \Rightarrow P) \tag{2}$$
$$\big(P \Rightarrow (Q \Rightarrow R)\big) \Rightarrow \big((P \Rightarrow Q) \Rightarrow (P \Rightarrow R\big) \tag{3}$$
$$(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P) \tag{4}$$

Here $\neg$ is the negation operator. $\neg P$ means *not* P.

- (8 Points) Give the truth table for each of the four axioms. Using this, should we accept the axioms as valid?
- (4 Points) The first axiom is redundant. Together with *modus ponens* (which is the axiom that if P implies Q is true, and if P is true, then Q is true), the second and third axiom can be used to prove that the first axiom is true. Prove this (partial credit will of course be given).

Problem 2 (Disjunction and Conjunction)

The *logical or* and *logical and* are not primitives, but rather can be defined with implication and negation. It is common to use the $\vee$ symbol for *or* and the $\wedge$ symbol for *and*. $P \vee Q$ then reads P *or* Q, and $P \wedge Q$ reads P *and* Q. These can be defined as follows:

$$(P \vee Q) \Leftrightarrow (\neg P \Rightarrow Q) \tag{5}$$
$$(P \wedge Q) \Leftrightarrow \neg(P \Rightarrow \neg Q) \tag{6}$$

Where $\Leftrightarrow$ means *is equivalent to* or *if and only if.*

- (2 Points) $P \vee Q$ is only false when both P and Q are false. Explain (with words, no mathematics needed here) when $\neg P \Rightarrow Q$ is false. Create the truth table for $\neg P \Rightarrow Q$ and explain why this is a valid choice for the logical or.

- (2 Points) $P \wedge Q$ is only true when both P and Q are true. Explain why $\neg(P \Rightarrow \neg Q)$ is a good choice for logical and. Construct the truth table for this.

- (6 Points) Prove that *or* is commutative. That is, $P \vee Q$ if and only if $Q \vee P$. You must prove:

$$(\neg P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow P) \tag{7}$$
$$(\neg Q \Rightarrow P) \Rightarrow (\neg P \Rightarrow Q) \tag{8}$$

  Hint: Use your Hilbert system.

Problem 3 (Set Arithmetic)

Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$. We use this often to prove two expressions are equal. Remember, $A \subseteq B$ if and only if $x \in A$ implies $x \in B$.

- (3 Points) Prove the distributive law of unions:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{9}$$

- (3 Points) Prove the distributive law of intersections:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{10}$$

- (3 Points) Prove De Morgan's Law of Unions. If $A, B \subseteq X$, then:

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \tag{11}$$

- (3 Points) Prove De Morgan's Law of Intersections. If $A, B \subseteq X$, then:

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) \tag{12}$$

Problem 4 (The Cantor-Schroeder-Bernstein Theorem)

There are two versions of the Cantor-Schroeder-Bernstein theorem. The first says that if A and B are sets, and if $f : A \to B$ and $g : B \to A$ are injective, then there is a bijection $h : A \to B$. The second states that if A and B are sets, and if $f : A \to B$ and $g : B \to A$ are surjective, then there is a bijection $h : A \to B$.

- (3 Points) Prove that if $f : A \to B$ is an injective function, then there is a surjection $g : B \to A$.

- (3 Points) Prove that if $f : A \to B$ is a surjective function, then there is an injection $g : B \to A$.

- (4 Points) Prove that the truth of the first Cantor-Schroeder-Bernstein theorem implies the validity of the second, and vice-versa.

Problem 5 (Induced Metrics)

A *norm* on $\mathbb{R}^n$ is a function that assigns a *length* to each point. That is, a function $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ such that for all points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all real numbers $a \in \mathbb{R}$ we have:

$$||\mathbf{x}|| \geq 0 \quad \text{(Positivity)}$$
$$||\mathbf{x}|| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \quad \text{(Definiteness)}$$
$$||a\mathbf{x}|| = |a| \cdot ||\mathbf{x}|| \quad \text{(Homogeneity)}$$
$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \quad \text{(Triangle-Inequality)}$$

The metric induced by a norm is:

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| \quad (13)$$

- (4 Points) Prove that the induced metric is a metric on $\mathbb{R}^n$.
- (6 Points) A convex set is a set $A \subseteq \mathbb{R}^n$ such that for all $\mathbf{x}, \mathbf{y} \in A$ and for all $0 \leq t \leq 1$ it is true that $t\mathbf{x} + (1-t)\mathbf{y} \in A$. Prove that open balls centered about the origin are convex when the metric comes from a norm.

Problem 6 (Connected Subsets)

(4 Points) A connected subset of a metric space (X, d) is a subset $A \subseteq X$ such that it is impossible to write $A = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are disjoint non-empty open sets. Give an example that shows that open balls do not need to be connected.