Point-Set Topology: Homework 2

Summer 2023

Problem 1 (Subspaces)

The inclusion mapping of a subset $A \subseteq X$ into X is the function $\iota_A : A \to X$ defined by $\iota_A(x) = x$.

- (1 Point) Prove that if (X, d) is a metric space, and if (A, d_A) is a metric subspace, then ι_A is continuous.
- (3 Points) Suppose (X, d_X) and (Y, d_Y) are metric spaces and let $A \subseteq X$ be a subset and d_A be the subspace metric. Prove that $f : Y \to A$ is continuous *if and only if* $\iota_A \circ f : Y \to X$ is continuous.
- (2 Points) Prove that if $f : X \to Y$ is a continuous function from a metric space (X, d_X) to a metric space (Y, d_Y), and if (A, d_A) is a subspace of (X, d_X), then the restriction $f|_A : A \to Y$, defined by $f|_A(x) = f(x)$, is continuous.
- (4 Points) Prove that if $f : X \to Y$ is a homeomorphism, and if $A \subseteq X$, then $f|_A : A \to f[A]$ is a homeomorphism.

Problem 2 (Continuity)

We have proven the equivalence of three definitions of continuity. The definition is that f maps convergent sequences to convergent sequences. The calculus $\varepsilon - \delta$ statement is equivalent to this, as is the fact that the pre-image of open sets is open. Continuity can be described by forward images as well.

- (6 Points) Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that $f : X \to Y$ is continuous if and only if for all $x \in X$ and for all open subsets $\mathcal{V} \subseteq Y$ with $f(x) \in \mathcal{V}$ there is an open subset $\mathcal{U} \subseteq X$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$.

Problem 3 (Compact Spaces)

For metric spaces there are many equivalent ways of defining compactness. Your job is to prove some of these equivalences.

- (4 Points) Prove that (X, d) is compact if and only if for every sequence of closed non-empty nested sets, the intersection is non-empty. That is, if $\mathcal{C} : \mathbb{N} \to \mathcal{P}(X)$ is a sequence of closed sets such that $\mathcal{C}_n \neq \emptyset$ and $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$, then $\bigcap_{n \in \mathbb{N}} \mathcal{C}_n$ is non-empty.

- (2 Points) Prove that (X, d) is compact if and only if for every sequence of nested proper open subsets, the union is not the whole space. That is, if $\mathcal{U} : \mathbb{N} \to \mathcal{P}(X)$ is a sequence of open sets such that $\mathcal{U}_n \neq X$ and $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$, then $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ is not equal to X. [Hint: What is the complement of an open set? Can the previous part of the problem help?]

Problem 4 (Calculus)

With our tools from metric space theory, one of the harder theorems from calculus becomes quite simple.

- (4 Points) Prove that if (X, d_X) is compact, if (Y, d_Y) is a metric space, and if $f : X \to Y$ is continuous, then $f[X] \subseteq Y$ is a compact subspace.

- (4 Points) The extreme value theorem states that if $f : [a, b] \to \mathbb{R}$ is continuous, then there is $c_{\min}, c_{\max} \in [a, b]$ such that $f(c_{\min}) \leq f(x) \leq f(c_{\max})$ for all $x \in [a, b]$. Let's take that up a notch. Prove that if (X, d) is compact, and if $f : X \to \mathbb{R}$ is continuous, then there are points $c_{\min}$ and $c_{\max}$ such that $f(c_{\min}) \leq f(x) \leq f(c_{\max})$ for all $x \in X$. [Hint: The previous part is enormously helpful.]

Problem 5 (Product Spaces)

(6 Points) Given metric spaces (X, d_X) and (Y, d_Y), prove that all three product metrics are topologically equivalent:

$$d_1\big((x_0, y_0), (x_1, y_1)\big) = d_X(x_0, x_1) + d_Y(y_0, y_1) \tag{1}$$

$$d_2\big((x_0, y_0), (x_1, y_1)\big) = \sqrt{d_X(x_0, x_1)^2 + d_Y(y_0, y_1)^2} \tag{2}$$

$$d_\infty\big((x_0, y_0), (x_1, y_1)\big) = \max\big(d_X(x_0, x_1), d_Y(y_0, y_1)\big) \tag{3}$$