Point-Set Topology: Homework 3

Summer 2023

Problem 1 (Separability)

A separable topological space is a space (X, τ) such that there is a countable subset $A \subseteq X$ such that $\mathrm{Cl}_\tau(A) = X$. A metric space is separable if and only if it is second-countable. This feature is special to metric spaces. Take $\mathbb{R}$ with the standard topology, and equip $\mathbb{R}/\mathbb{Z}$ with the quotient topology. Intuitively this is infinite many circles all touching at 0. It is not first-countable, and hence not second-countable, even though $\mathbb{R}$ is. It is still separable.

- (6 Points) Let (X, τ) be a separable topological space. Let R be any equivalence relation on X. Prove that $(X/R, \tau_{X/R})$ is separable. That is, separability is a topological property preserved by quotients.

Problem 2 (Embeddings)

The bug-eyed line is a quotient space of $X = \mathbb{R} \times \{0, 1\}$ where $\mathbb{R}$ has the standard Euclidean topology and $\{0, 1\}$ has the discrete topology. X is given the product topology. We identity $(x, 0)$ with $(x, 1)$ for all $x \neq 0$ and then take the quotient of X under this relation. This idea is shown in Fig. 1

- (6 Points) Prove that it is impossible to embed the bug-eyed line into $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

Problem 3 (Quotients)

Let $X = \mathbb{R}/\mathbb{Q}$, equipped with the quotient topology where $\mathbb{R}$ carries the usual Euclidean topology.

- (4 Points) Is this space Hausdorff? Is it Fréchet?

Problem 4 (Products)

Consider topological spaces (X, τ_X), (Y, τ_Y), and (Z, τ_Z). Equip $X \times Y$ with the product topology $\tau_{X\times Y}$.

- (6 Points) Prove that a function $f : Z \to X \times Y$ is continuous if and only if the component functions $\mathrm{proj}_X \circ f : Z \to X$ and $\mathrm{proj}_Y \circ f : Z \to Y$ are continuous.

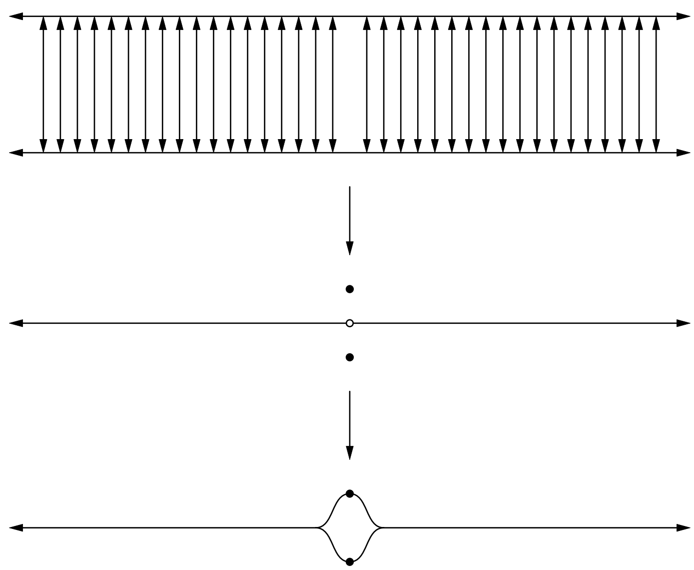

Figure 1: The Bug-Eyed Line Construction