Point-Set Topology: Homework 4

Summer 2023

Problem 1 (Comparing Topologies)

Let X be a set, and τ and τ' topologies on X with $\tau \subseteq \tau'$.

1. (2 Pts) Prove that if (X, τ) is Fréchet, then (X, τ') is.
2. (2 Pts) Prove that if (X, τ) is Hausdorff, then (X, τ') is.
3. (3 Pts) If (X, τ) is regular, does this imply (X, τ') is? Prove this or provide a counterexample.
4. (3 Pts) If (X, τ) is normal, does this imply (X, τ') is? Prove this or provide a counterexample.

Problem 2 (The Sorgenfrey Line)

The Sorgenfrey line is the real line $\mathbb{R}$ with the left-interval topology. That is, the topology is generated by sets of the form $[a, b)$ for $a, b \in \mathbb{R}$.

1. (3 Pts) A zero dimensional space is a topological space (X, τ) such that there is a basis $\mathcal{B}$ of sets that are both open and closed. Prove that the Sorgenfrey line is zero dimensional. (Hint: Prove $[a, b)$ is open and closed in this topology).
2. (5 Pts) Prove that a zero dimensional topological space is completely regular.
3. (2 Pts) Letting $\tau_{\mathbb{R}}$ and τ_S be the Euclidean and Sorgenfrey topologies, respectively, on $\mathbb{R}$, prove that $\tau_{\mathbb{R}} \subseteq \tau_S$. [Hint: What is a basis of $\tau_{\mathbb{R}}$? Are these elements open in τ_S?]
4. (8 Pts) Prove that the Sorgenfrey line is Lindelöf. That is, every open cover has a countable subcover. Do this by showing that any open cover of the Sorgenfrey line by basis elements (sets of the form $[a, b)$) has a countable subcover. [Hint: Sets of the form (a, b) are open in the Euclidean topology. Since the Euclidean topology is second-countable, every subspace is Lindelöf. Prove that, if $\mathcal{O}$ is an open cover of the Sorgenfrey line of basis elements $[a, b)$, then the sets of the form (a, b) cover all but a countable subset of $\mathbb{R}$].

Problem 3 (The Sorgenfrey Plane)

A regular Lindelöf space is paracompact, meaning the Sorgenfrey line is a paracompact Hausdorff space. By Dieudonné's theorem it is therefore normal. Here you will prove that normality is not preserved by products.

1. (3 Pts) Prove that the product of two Hausdorff spaces is Hausdorff.
2. (3 Pts) Prove that the product of two regular spaces is regular.
3. (5 Pts) The Sorgenfrey Plane is the product of the Sorgenfrey line with itself. The anti-diagonal:

$$\Delta = \{(x,\, -x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \tag{1}$$

   has the strange property that it is a discrete subspace. That is, the subspace topology of Δ is the power set of Δ. Prove this.

4. (5 Pts) The set K of points in Δ with rational coordinates:

$$K = \{\, (x,\, -x) \in \mathbb{R}^2 \mid x \in \mathbb{Q}\,\} \tag{2}$$

   is a closed subset of a closed subspace, so it too is closed. As is the set $\Delta \setminus K$. These sets cannot be separated by open sets, showing us that the Sorgenfrey plane is not normal. Prove that the Sorgenfrey plane is not paracompact, not Lindelof, and not second-countable.

5. (4 Pts) Prove that a closed subspace of a separable space need not be separable.

Problem 4 (Manifolds)

A locally-Euclidean topological space is a topological space $(X,\, \tau)$ such that for all $x \in X$ there is an open set $\mathcal{U} \in \tau$ with $x \in \mathcal{U}$ and a continuous injective open mapping $\varphi : \mathcal{U} \to \mathbb{R}^n$ for some $n \in \mathbb{N}$. A topological manifold is a topological space $(X,\, \tau)$ that is:

1. Hausdorff.
2. Second-Countable.
3. Locally-Euclidean.

The classic examples are Euclidean spaces $\mathbb{R}^n$, the circle, sphere, and higher-dimensional analogues $\mathbb{S}^n$, the Klein bottle, and the projective spaces $\mathbb{RP}^n$. The Hausdorff and second-countable conditions are not redundant. The bug-eyed line is second-countable and locally-Euclidean, but non-Hausdorff. The long line is Hausdorff and locally-Euclidean, but not second-countable.

You may use the fact that a locally-compact Hausdorff space is regular, should you find it useful.

1. (5 Pts) Recall that locally compact means for all x there is an open set $\mathcal{U}$ and a compact subset K such that $x \in \mathcal{U}$ and $\mathcal{U} \subseteq K$. Prove that a topological manifold is locally compact.
2. (3 Pts) Prove that a topological manifold is σ-compact.
3. (3 Pts) Prove that a topological manifold is paracompact. [Hint: Use your theorems. Not much work needed here.]
4. (3 Pts) Prove that a topological manifold is metrizable. [Hint: Again, appeal to theorems. Piece together the puzzle, not much effort should be applied here.]