Point-Set Topology: Homework 1

Summer 2023

Problem 1 (Hilbert Systems)

The Hilbert System is a collection of axioms for how propositional logic should behave. It claims the following four statements are true and do not need proof. Let P, Q, and R be propositions (statements that are true or false). Then the following are true:

$$P \Rightarrow P \tag{1}$$

$$P \Rightarrow (Q \Rightarrow P) \tag{2}$$

$$\big(P \Rightarrow (Q \Rightarrow R)\big) \Rightarrow \big((P \Rightarrow Q) \Rightarrow (P \Rightarrow R\big) \tag{3}$$

$$(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P) \tag{4}$$

Here $\neg$ is the negation operator. $\neg P$ means *not* P.

- (8 Points) Give the truth table for each of the four axioms. Using this, should we accept the axioms as valid?
- (4 Points) The first axiom is redundant. Together with *modus ponens* (which is the axiom that if P implies Q is true, and if P is true, then Q is true), the second and third axiom can be used to prove that the first axiom is true. Prove this (partial credit will of course be given).

Solution. The truth table for $P \Rightarrow P$ is given in Tab. 1. In general, $P \Rightarrow Q$ is only false when P is true and Q is false. So $P \Rightarrow P$ would only be false when P is true and P is false, a contradiction. So $P \Rightarrow P$ is always true, unless there exists a contradiction in our language (hopefully there doesn't).

The truth table for Hilbert's second axiom, $P \Rightarrow (Q \Rightarrow P)$, is given in Tab. 2. Again, $P \Rightarrow Q$ is false only when P is true and Q is false. So $P \Rightarrow (Q \Rightarrow P)$ is

P	$P \Rightarrow P$
False	True
True	True

Table 1: Truth Table for Hilbert's First Axiom

P	Q	$Q \Rightarrow P$	$P \Rightarrow (Q \Rightarrow P)$
False	False	True	True
False	True	False	True
True	False	True	True
True	True	True	True

Table 2: Truth Table for Hilbert's Second Axiom

P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$(P \Rightarrow Q) \Rightarrow$ $(P \Rightarrow R)$	$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow$ $((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$
False	False	False	True	True	True	True	True	True
False	False	True	True	True	True	True	True	True
False	True	False	True	True	False	True	True	True
False	True	True	True	True	True	True	True	True
True	False	False	False	False	True	True	True	True
True	False	True	False	True	True	True	True	True
True	True	False	True	False	False	False	False	True
True	True	True	True	True	True	True	True	True

Table 3: Truth Table for Hilbert's Third Axiom

false only when P is true and $Q \Rightarrow P$ is false. Examining $Q \Rightarrow P$, this is only false when Q is true and P is false. So if P is true, $Q \Rightarrow P$ is true, meaning $P \Rightarrow (Q \Rightarrow P)$ is also true.

The massive truth table for Hilbert's third axiom is given in Tab. 3. The only case to inspect is when $P \Rightarrow (Q \Rightarrow R)$ is true and $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ is false. In this case P must be true, otherwise $P \Rightarrow Q$ would true, and $P \Rightarrow R$ would be true, and hence $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ would be true. But since $P \Rightarrow (Q \Rightarrow R)$ is true, and since P is true, $Q \Rightarrow R$ must be true. Now Q must be true, for if not $P \Rightarrow Q$ would be false, since P is true, and then $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ would be true. We have concluded thus far that P and Q are true, so $P \Rightarrow Q$ is true. But $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ is supposed to be false, and thus $P \Rightarrow R$ must be false. And since P is true, R must be false. That is, P is true, Q is true, and R is false. We may thus conclude that $Q \Rightarrow R$ is false. But $P \Rightarrow (Q \Rightarrow R)$ is true, and P is true, meaning $Q \Rightarrow R$ is true, a contradiction. So the scenario that $P \Rightarrow (Q \Rightarrow R)$ is true and $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ is false never occurs, unless we have a contradiction.

The final truth table is given in Tab. 4. This is the *law of the contrapositive*. Note

P	Q	$\neg P$	$\neg Q$	$\neg P \Rightarrow \neg Q$	$Q \Rightarrow P$	$(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow R)$
False	False	True	True	True	True	True
False	True	True	False	False	False	True
True	False	False	True	True	True	True
True	True	False	False	True	True	True

Table 4: Truth Table for Hilbert's Fourth Axiom

that $\neg P \Rightarrow \neg Q$ and $Q \Rightarrow P$ have identical columns. This is because they are equivalent statements. Mathematicians often use the law of the contrapositive to prove statements $P \Rightarrow Q$ when $\neg Q \Rightarrow \neg P$ is easier.

This is not an *independent* system of axioms, the first statement can be proved from axioms 2 and 3 (together with *modus ponens*). By setting $R = P$ and $Q = (P \Rightarrow P)$ we obtain:

$$\begin{aligned}
\big(P \Rightarrow (Q \Rightarrow R)\big) \Rightarrow \big((P \Rightarrow Q) \Rightarrow (P \Rightarrow R)\big) && \text{(Axiom 3)}\\
P \Rightarrow (Q \Rightarrow P) && \text{(Axiom 2)}\\
\Big(P \Rightarrow \big((P \Rightarrow P) \Rightarrow P\big)\Big) \Rightarrow \Big(\big(P \Rightarrow (P \Rightarrow P)\big) \Rightarrow (P \Rightarrow P)\Big) && \text{(Substitute)}\\
P \Rightarrow \big((P \Rightarrow P) \Rightarrow P\big) && \text{(Substitute)}\\
\big(P \Rightarrow (P \Rightarrow P)\big) \Rightarrow (P \Rightarrow P) && \text{(Modus Ponens)}\\
P \Rightarrow (P \Rightarrow P) && \text{(Axiom 2)}\\
P \Rightarrow P && \text{(Modus Ponens)}
\end{aligned}$$

So the negation of the first axiom would be *inconsistent* with the others. □

Problem 2 (Disjunction and Conjunction)

The *logical or* and *logical and* are not primitives, but rather can be defined with implication and negation. It is common to use the $\vee$ symbol for *or* and the $\wedge$ symbol for *and*. $P \vee Q$ then reads P *or* Q, and $P \wedge Q$ reads P *and* Q. These can be defined as follows:

$$(P \vee Q) \Leftrightarrow (\neg P \Rightarrow Q) \tag{5}$$

$$(P \wedge Q) \Leftrightarrow \neg(P \Rightarrow \neg Q) \tag{6}$$

Where $\Leftrightarrow$ means *is equivalent to* or *if and only if*.

- (2 Points) $P \vee Q$ is only false when both P and Q are false. Explain (with words, no mathematics needed here) when $\neg P \Rightarrow Q$ is false. Create the truth table for $\neg P \Rightarrow Q$ and explain why this is a valid choice for the logical or.
- (2 Points) $P \wedge Q$ is only true when both P and Q are true. Explain why $\neg(P \Rightarrow \neg Q)$ is a good choice for logical and. Construct the truth table for this.
- (6 Points) Prove that *or* is commutative. That is, $P \vee Q$ if and only if $Q \vee P$. You must prove:

  $$(\neg P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow P) \tag{7}$$

  $$(\neg Q \Rightarrow P) \Rightarrow (\neg P \Rightarrow Q) \tag{8}$$

  Hint: Use your Hilbert system.

Solution. $P \Rightarrow Q$ is only false when P is true, yet Q is false. Introducing negation, $\neg P \Rightarrow Q$ is false when $\neg P$ is true and Q is false. $\neg P$ being true means P is false, and hence $\neg P \Rightarrow Q$ is false only when P and Q are both false. This is the same condition for logical or, meaning it is a good candidate for the definition of $P \vee Q$. The truth table is given in Tab. 5.

Logical *and*, or conjunction, is only true when both propositions are true. $P \Rightarrow \neg Q$ is only false when P and Q are both true, meaning $\neg(P \Rightarrow \neg Q)$ is only true when both P and Q are true. The truth table is given in Tab. 6.

We can use the Hilbert system to prove some of the basic laws of logical *or* and *and*, such as commutativity, associativity, and much more. To prove commutativity we wish to show that:

$$P \vee Q \Leftrightarrow Q \vee P \tag{9}$$

which is equivalent to two implications:

$$P \vee Q \Rightarrow Q \vee P \tag{10}$$

$$Q \vee P \Rightarrow P \vee Q \tag{11}$$

P	Q	$\neg P$	$\neg P \Rightarrow Q$	$P \vee Q$
False	False	True	False	False
False	True	True	True	True
True	False	False	True	True
True	True	False	True	True

Table 5: Truth Table for Disjunction

P	Q	$\neg Q$	$\neg(P \Rightarrow \neg Q)$	$P \wedge Q$
False	False	True	False	False
False	True	False	False	False
True	False	True	False	False
True	True	False	True	True

Table 6: Truth Table for Conjunction

Since we have defined disjunction using implication and negation, we can expand the $\vee$ symbol out and get the following implications:

$$(\neg P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow P) \tag{12}$$

$$(\neg Q \Rightarrow P) \Rightarrow (\neg P \Rightarrow Q) \tag{13}$$

Let's prove these two claims. Hilbert's fourth says $(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)$. Substituting $P = P$ and $Q = \neg Q$, and invoking the law of double negation $(Q \Leftrightarrow \neg\neg Q)$, we get:

$$(\neg P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow P) \tag{14}$$

Which is the first desired implication. Returning to Hilbert's fourth axiom, $(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)$, if we substitute $P = Q$ and $Q = \neg P$, we get:

$$(\neg Q \Rightarrow P) \Rightarrow (\neg P \Rightarrow Q) \tag{15}$$

which is the second desired implication. Hence $P \vee Q \Rightarrow Q \vee P$ and $Q \vee P \Rightarrow P \vee Q$. That is, the logical *or* is commutative. □

Problem 3 (Set Arithmetic)

Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$. We use this often to prove two expressions are equal. Remember, $A \subseteq B$ if and only if $x \in A$ implies $x \in B$.

- (3 Points) Prove the distributive law of unions:
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{16}$$
- (3 Points) Prove the distributive law of intersections:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{17}$$
- (3 Points) Prove De Morgan's Law of Unions. If $A, B \subseteq X$, then:
$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \tag{18}$$
- (3 Points) Prove De Morgan's Law of Intersections. If $A, B \subseteq X$, then:
$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) \tag{19}$$

Solution. Let's start with $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. We do this by showing the left-hand side is a subset of the right-hand side, and vice-versa. Suppose $x \in A \cup (B \cap C)$. Then, by definition of union, $x \in A$ or $x \in B \cap C$. Then $x \in A$ or $x \in B$ and $x \in C$. But if $x \in A$, then $x \in A$ or $x \in B$, by definition of *or*. Similarly if $x \in A$, then $x \in A$ or $x \in C$. Hence if $x \in A$ then $x \in A$ or B and $x \in A$ or C, so $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and C by definition of intersection. Again by our use of the word *or*, $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$, so $x \in (A \cup B) \cap (A \cup C)$, meaning $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. In the other direction, if $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$. So either $x \in A$ is true, or $x \in B$ and $x \in C$ is true, meaning $x \in A \cup (B \cap C)$, and hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

For the next equality we need to show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Suppose $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$, by the definition of intersection. But then $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$, by definition of union. Hence $x \in A \cap B$ or $x \in A \cap C$, meaning $x \in (A \cap B) \cup (A \cap C)$. In the other direction, if $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap B$ or $x \in A \cap C$. But then $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$, by definition of intersection. This means $x \in A$ and either $x \in B$ or $x \in C$, meaning $x \in A \cap (B \cup C)$.

Now for the De Morgan laws. Suppose $x \in X \setminus (A \cup B)$. Then $x \notin A \cup B$, which means $x \notin A$ and $x \notin B$. That is, $x \in (X \setminus A) \cap (X \setminus B)$, so $X \setminus (A \cup B) \subseteq$

$(X \setminus A) \cap (X \setminus B)$. Now suppose $x \in (X \setminus A) \cap (X \setminus B)$. Then $x \in X \setminus A$ and $x \in X \setminus B$. That is, $x \notin A$ and $x \notin B$, and hence $x \notin A \cup B$. But if $x \notin A \cup B$ and $x \in X$, then $x \in X \setminus (A \cup B)$. Therefore $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$, and thus $(X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B)$.

The other De Morgan law is dealt with similarly. Suppose $x \in X \setminus (A \cap B)$. Then $x \notin A \cap B$, meaning $x \notin A$ or $x \notin B$. But then $x \in (X \setminus A) \cup (X \setminus B)$, implying $X \setminus (A \cap B) \subseteq (X \setminus A) \cup (X \setminus B)$. Now suppose $x \in (X \setminus A) \cup (X \setminus B)$. Then $x \notin A$ or $x \notin B$. But then $x \notin A \cap B$, and therefore $x \in X \setminus (A \cap B)$. We conclude that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. $\square$

Problem 4 (The Cantor-Schroeder-Bernstein Theorem)

There are two versions of the Cantor-Schroeder-Bernstein theorem. The first says that if A and B are sets, and if $f : A \to B$ and $g : B \to A$ are injective, then there is a bijection $h : A \to B$. The second states that if A and B are sets, and if $f : A \to B$ and $g : B \to A$ are surjective, then there is a bijection $h : A \to B$.

- (3 Points) Prove that if $f : A \to B$ is an injective function, then there is a surjection $g : B \to A$.
- (3 Points) Prove that if $f : A \to B$ is a surjective function, then there is an injection $g : B \to A$.
- (4 Points) Prove that the truth of the first Cantor-Schroeder-Bernstein theorem implies the validity of the second, and vice-versa.

Solution. The problem is *vacuous* if A or B are empty, so first suppose they are not. Since A is non-empty, pick some $x \in A$. Given an injective function $f : A \to B$, define $g : B \to A$ as follows:

$$g(b) = \begin{cases} a \in A \text{ such that } f(a) = b \\ x \text{ if such an element does not exist} \end{cases} \tag{20}$$

There's no axiom of choice needed here, since f is injective we have a well-defined function. g is surjective. Given $a \in A$ let $b = f(a)$. Then $g(b) = a$ by definition.

Given a surjection $f : A \to B$, for each $b \in B$ pick some $a_b \in A$ such that $f(a_b) = b$ and define $g(b) = a_b$. This *choosing* does invoke the axiom of choice, but very subtly. The function g is an injective function from B to A since $g(b_0) = g(b_1)$ implies $a_{b_0} = a_{b_1}$, which means:

$$b_0 = f(a_{b_0}) = f(a_{b_1}) = b_1 \tag{21}$$

and hence $b_0 = b_1$, so g is injective.

Now suppose the first Cantor-Schroeder-Bernstein theorem is true. That is, if there exists injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$. Let's use this to prove the second Cantor-Schroeder-Bernstein theorem. Suppose $\tilde{f} : A \to B$ and $\tilde{g} : B \to A$ are surjective functions. By the previous part of the problem there then exists injective functions $f : A \to B$ and $g : B \to A$. But then by the first Cantor-Schroeder-Bernstein theorem there is a bijection $h : A \to B$. Hence the existence of surjective functions $\tilde{f} : A \to B$ and $\tilde{g} : B \to B$ implies the existence of a bijection $h : A \to B$.

In the other direction, suppose the second Cantor-Schroeder-Bernstein theorem holds. That is, if $f : A \to B$ and $g : B \to A$ are surjective functions, then there is a bijection $h : A \to B$. Given injective function $\tilde{f} : A \to B$ and $\tilde{g} : B \to A$, by the previous part of the problem there are surjective functions $f : A \to B$ and $g : B \to A$. Then by the second Cantor-Schroeder-Bernstein theorem there is a bijection $h : A \to B$. Hence the existence of injective functions $\tilde{f} : A \to B$ and $\tilde{g} : B \to A$ implies the existence of a bijection $h : A \to B$.

Since the first Cantor-Schroeder-Bernstein theorem implies the second, and vice-versa, all you need to do is prove *one* of them, and the second immediately follows. Most texts usually prove the first one, that if there exists injective function $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$. $\square$

Problem 5 (Induced Metrics)

A *norm* on $\mathbb{R}^n$ is a function that assigns a *length* to each point. That is, a function $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ such that for all points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all real numbers $a \in \mathbb{R}$ we have:

$$||\mathbf{x}|| \geq 0 \qquad \text{(Positivity)}$$
$$||\mathbf{x}|| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \qquad \text{(Definiteness)}$$
$$||a\mathbf{x}|| = |a| \cdot ||\mathbf{x}|| \qquad \text{(Homogeneity)}$$
$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \qquad \text{(Triangle-Inequality)}$$

The metric induced by a norm is:

$$d(\mathbf{x},\, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| \tag{22}$$

- (4 Points) Prove that the induced metric is a metric on $\mathbb{R}^n$.
- (6 Points) A convex set is a set $A \subseteq \mathbb{R}^n$ such that for all $\mathbf{x}, \mathbf{y} \in A$ and for all $0 \leq t \leq 1$ it is true that $t\mathbf{x} + (1-t)\mathbf{y} \in A$. Prove that open balls centered about the origin are convex when the metric comes from a norm.

Solution. This function is indeed a metric. It is positive-definite since $||\mathbf{x}||$ is non-negative and:

$$d(\mathbf{x},\, \mathbf{y}) = 0 \tag{23}$$
$$\Leftrightarrow ||\mathbf{x} - \mathbf{y}|| = 0 \tag{24}$$
$$\Leftrightarrow \mathbf{x} - \mathbf{y} = \mathbf{0} \tag{25}$$
$$\Leftrightarrow \mathbf{x} = \mathbf{y} \tag{26}$$

It is symmetric since:

$$d(\mathbf{x},\, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| \tag{27}$$
$$= ||(-1)(\mathbf{y} - \mathbf{x})|| \tag{28}$$
$$= |-1|\; ||\mathbf{y} - \mathbf{x}|| \tag{29}$$
$$= ||\mathbf{y} - \mathbf{x}|| \tag{30}$$
$$= d(\mathbf{y},\, \mathbf{x}) \tag{31}$$

Lastly, the triangle inequality is satisfied. We have, for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$:

$$d(\mathbf{x},\, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| \tag{32}$$
$$= ||\mathbf{x} + \mathbf{0} - \mathbf{y}|| \tag{33}$$
$$= ||\mathbf{x} + (-\mathbf{z} + \mathbf{z}) - \mathbf{y}|| \tag{34}$$
$$= ||(\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{y})|| \tag{35}$$
$$\leq ||\mathbf{x} - \mathbf{z}|| + ||\mathbf{z} - \mathbf{y}|| \tag{36}$$
$$= ||\mathbf{x} - \mathbf{z}|| + |\mathbf{y} - \mathbf{z}|| \tag{37}$$
$$= d(\mathbf{x},\, \mathbf{z}) + d(\mathbf{y},\, \mathbf{z}) \tag{38}$$

So d is a metric.

The balls in a normed vector space are convex. Since translation $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by $T(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ for some fixed $\mathbf{a} \in \mathbb{R}^n$, is a global isometry, and since $||a\mathbf{x}|| = |a| \, ||\mathbf{x}||$ for all real $a \in \mathbb{R}$, we can consider open balls of radius 1 centered at the origin. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ be such that $d(\mathbf{x}, \mathbf{0}) < 1$ and $d(\mathbf{y}, \mathbf{0}) < 1$. In other words, suppose $||\mathbf{x}|| < 1$ and $||\mathbf{y}|| < 1$. Then for any $0 \leq t \leq 1$ we have:

$$\begin{aligned}
||t\mathbf{x} + (1-t)\mathbf{y}|| &\leq ||t\mathbf{x}|| + ||(1-t)\mathbf{y}|| && \text{(Triangle Inequality)} \\
&= |t| \, ||\mathbf{x}|| + |1-t| \, ||\mathbf{y}|| && \text{(Factoring Scalars)} \\
&< |t| + |1-t| && \text{(Since } ||\mathbf{x}|| < 1 \text{ and } ||\mathbf{y}|| < 1) \\
&= t + 1 - t && \text{(Since } 0 \leq t \leq 1) \\
&= 1 && \text{(Simplify)}
\end{aligned}$$

And hence this point lies in the unit ball as well. $\square$

Problem 6 (Connected Subsets)

(4 Points) A connected subset of a metric space (X, d) is a subset $A \subseteq X$ such that it is impossible to write $A = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are disjoint non-empty open sets. Give an example that shows that open balls do not need to be connected.

Solution. There are many examples, but the easiest is probably the discrete metric on a two point set. Let $X = \{\, 0,\, 1\,\}$ and d be the discrete metric on X. The ball of radius 2 centered about zero is disconnected since it can be written as the union of the ball of radius 1 about 0 and the ball of radius 1 about 1, two non-empty disjoint open sets. That is, the set $\{\, 0,\, 1\,\}$, which is the open ball of radius 2 centered at 0, can be written as the union of $\{\, 0\,\}$ and $\{\, 1\,\}$, which are the open balls of radius 1 centered about 0 and 1, respectively. $\square$