Point-Set Topology: Homework 2

Summer 2023

Problem 1 (Subspaces)

The inclusion mapping of a subset $A \subseteq X$ into X is the function $\iota_A : A \to X$ defined by $\iota_A(x) = x$.

- (1 Point) Prove that if $(X,\, d)$ is a metric space, and if $(A,\, d_A)$ is a metric subspace, then ι_A is continuous.
- (3 Points) Suppose $(X,\, d_X)$ and $(Y,\, d_Y)$ are metric spaces and let $A \subseteq X$ be a subset and d_A be the subspace metric. Prove that $f : Y \to A$ is continuous *if and only if* $\iota_A \circ f : Y \to X$ is continuous.
- (2 Points) Prove that if $f : X \to Y$ is a continuous function from a metric space $(X,\, d_X)$ to a metric space $(Y,\, d_Y)$, and if $(A,\, d_A)$ is a subspace of $(X,\, d_X)$, then the restriction $f|_A : A \to Y$, defined by $f|_A(x) = f(x)$, is continuous.
- (4 Points) Prove that if $f : X \to Y$ is a homeomorphism, and if $A \subseteq X$, then $f|_A : A \to f[A]$ is a homeomorphism.

Solution. Let $a : \mathbb{N} \to A$ be a convergent sequence with $a_n \to x$ for some $x \in A$. That is, $d_A(a_n,\, x) \to 0$. But then:

$$\begin{aligned} d\big(\iota_A(a_n),\, \iota_A(x)\big) &= d_A\big(\iota_A(a_n),\, \iota_A(x)\big) && \text{(Definition of } d_A\text{)} \\ &= d_A(a_n,\, x) && \text{(Definition of } \iota_A\text{)} \end{aligned}$$

and hence $d(\iota_A(a_n),\, \iota_A(x)) \to 0$, so $\iota_A(a_n) \to \iota_A(x)$, meaning ι_A is continuous.

Suppose $f : Y \to A$ is continuous. Then since $\iota_A : A \to X$ is continuous, $\iota_A \circ f$ is the composition of continuous functions, which is continuous. In the other direction, suppose $\iota_A \circ f$ is continuous. Let $a : \mathbb{N} \to Y$ be a convergent sequence with limit $y \in Y$. We must prove that $f(a_n) \to f(y)$. Let $\varepsilon > 0$ be given. Since, by hypothesis, $\iota_A \circ f$ is continuous, there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $d\big((\iota_A \circ f)(a_n),\, (\iota_A \circ f)(y)\big) < \varepsilon$. But then, by definition of d_A and ι_A, we have that $n > N$ implies $d_A\big(f(a_n),\, f(y)\big) < \varepsilon$, and therefore $f(a_n) \to f(y)$. That is, $f : Y \to A$ is continuous.

For part 3, let $a : \mathbb{N} \to A$ be a convergent sequence with limit $x \in A$. We must prove $f|_A(a_n) \to f|_A(x)$. Let $\varepsilon > 0$. Since $A \subseteq X$ and $a : \mathbb{N} \to A$ is a convergent sequence in A, $a : \mathbb{N} \to X$ is a convergent sequence in X as well with the same limit. But $f : X \to Y$ is continuous, so $f(a_n) \to f(x)$. But then, since $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $d_Y\big(f(a_n),\, f(x)\big) < \varepsilon$. But $f|_A(a_n) = f(a_n)$ and $f|_A(x) = f(x)$, so $n > N$ implies $d_Y\big(f|_A(a_n),\, f|_A(x)\big) < \varepsilon$. Hence $f|_A(a_n) \to f|_A(x)$ and $f|_A$ is continuous.

Lastly, we are to prove the restriction of a homeomorphism to a subspace yields a homeomorphism to the image. The restriction is continuous by the previous problem. Since $f : X \to Y$ is bijective, it is injective, and hence $f|_A$ is injective as well. Since the co-domain is $f[A]$, the function is also surjective. So $f|_A : A \to f[A]$ is a continuous bijection. We must prove the inverse function is continuous. But the inverse function $(f|_A)^{-1}$ is $f^{-1}|_{f[A]}$, the restriction of f^{-1} to $f[A]$. But f^{-1} is continuous since f is a homeomorphism. But then $f^{-1}|_{f[A]}$ is the restriction of a continuous function to a subspace, meaning it is continuous. Therefore $(f|_A)^{-1}$ is continuous, and $f|_A : A \to f[A]$ is a homeomorphism. $\square$

Problem 2 (Continuity)

We have proven the equivalence of three definitions of continuity. The definition is that f maps convergent sequences to convergent sequences. The calculus $\varepsilon - \delta$ statement is equivalent to this, as is the fact that the pre-image of open sets is open. Continuity can be described by forward images as well.

- (6 Points) Let $(X,\, d_X)$ and $(Y,\, d_Y)$ be metric spaces. Prove that $f : X \to Y$ is continuous if and only if for all $x \in X$ and for all open subsets $\mathcal{V} \subseteq Y$ with $f(x) \in \mathcal{V}$ there is an open subset $\mathcal{U} \subseteq X$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$.

Solution. It is easiest to prove this using the fact that a function is continuous if and only if the pre-image of an open set is open. First suppose $f : X \to Y$ is continuous. Let $x \in X$ and let $\mathcal{V} \subseteq Y$ be an open set such that $f(x) \in \mathcal{V}$. Since f is continous we have that $\mathcal{U} = f^{-1}[\mathcal{V}]$ is open. But then $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$, by definition of images and pre-images. Hence a continuous function has the desired property.

In the other direction, suppose $f : X \to Y$ is such that for all $x \in X$ and for all open sets $\mathcal{V} \subseteq Y$ with $f(x) \in \mathcal{V}$ there exists an open set $\mathcal{U} \subseteq X$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$. We must prove that the pre-image of an open set is open in order to conclude that f is continuous. Let $\mathcal{V} \subseteq Y$ be open. If $\mathcal{V}$ is empty we are done, since $f^{-1}[\emptyset] = \emptyset$. Similarly, if $f^{-1}[\mathcal{V}] = \emptyset$ there is nothing to prove. So suppose $\mathcal{V} \subseteq Y$ is an open set whose pre-image is not empty. Let $x \in f^{-1}[\mathcal{V}]$. Then, by definition of pre-image, $f(x) \in \mathcal{V}$. By hypothesis there is then open open set $\mathcal{U} \subseteq X$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$. But if $\mathcal{U}$ is open and $x \in \mathcal{U}$, then there is an $\varepsilon > 0$ such that $B_{\varepsilon}^{(X,\, d_X)}(x) \subseteq \mathcal{U}$. But then $B_{\varepsilon}^{(X,\, d_X)}(x) \subseteq f^{-1}[\mathcal{V}]$. That is, for all $x \in f^{-1}[\mathcal{V}]$ there is an ε ball about x contained entirely inside of $f^{-1}[\mathcal{V}]$, and hence $f^{-1}[\mathcal{V}]$ is open. Therefore f is continuous. $\square$

Problem 3 (Compact Spaces)

For metric spaces there are many equivalent ways of defining compactness. Your job is to prove some of these equivalences.

- (4 Points) Prove that $(X,\, d)$ is compact if and only if for every sequence of closed non-empty nested sets, the intersection is non-empty. That is, if $\mathcal{C}:\mathbb{N}\to\mathcal{P}(X)$ is a sequence of closed sets such that $\mathcal{C}_n\neq\emptyset$ and $\mathcal{C}_{n+1}\subseteq\mathcal{C}_n$, then $\bigcap_{n\in\mathbb{N}}\mathcal{C}_n$ is non-empty.
- (2 Points) Prove that $(X,\, d)$ is compact if and only if for every sequence of nested proper open subsets, the union is not the whole space. That is, if $\mathcal{U}:\mathbb{N}\to\mathcal{P}(X)$ is a sequence of open sets such that $\mathcal{U}_n\neq X$ and $\mathcal{U}_n\subseteq\mathcal{U}_{n+1}$, then $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ is not equal to X. [Hint: What is the complement of an open set? Can the previous part of the problem help?]

Solution. There are a few ways to prove this. Let's use sequences first. Suppose $(X,\, d)$ is compact and $\mathcal{C}:\mathbb{N}\to\mathcal{P}(X)$ is a sequence of nested closed non-empty sets. Then for each $n\in\mathbb{N}$, since $\mathcal{C}_n$ is non-empty, there is an $a_n\in\mathcal{C}_n$. Since $(X,\, d)$ is compact and $a:\mathbb{N}\to X$ is a sequence in X, there is a convergent subsequence a_k. Let $x\in X$ be the limit. Then for all $n\in\mathbb{N}$, $x\in\mathcal{C}_n$. To see this, given $N\in\mathbb{N}$, for all $n>N$ we have $a_{k_n}\in\mathcal{C}_{k_n}$, and since $k_n>N$ this implies $a_{k_n}\in\mathcal{C}_N$ since the sets are nested. So a_k is convergent sequence that is eventually contained in $\mathcal{C}_n$, and $\mathcal{C}_n$ is closed so it contains its limit points, and therefore $x\in\mathcal{C}_n$. Since this is true for all $n\in\mathbb{N}$, we have that $x\in\bigcap_{n\in\mathbb{N}}\mathcal{C}_n$. That is, the intersection is non-empty.

Now suppose $(X,\, d)$ is a metric space with the property that for all nested sequences of non-empty closed sets $\mathcal{C}:\mathbb{N}\to\mathcal{P}(X)$ it is true that $\bigcap_{n\in\mathbb{N}}\mathcal{C}_n$ is non-empty. Let us prove that $(X,\, d)$ is compact. Suppose not. Then there is a sequence $a:\mathbb{N}\to X$ with no convergent subsequence. Then the set:

$$A=\{a_n\in X\mid n\in\mathbb{N}\}\tag{1}$$

is closed. For if not then there is a point $x\in X$ that is a limit point of this set, but is not contained in it. But if x is a limit point of A then there is a sequence of points in this set that converges to x. But then the sequence $a:\mathbb{N}\to X$ would have a convergent subsequence, contradicting the claim that no such subsequence exists. Hence $A=\{\,a_n\in X\mid n\in\mathbb{N}\,\}$ is closed.

Alternatively, if you'd rather, we can show $X\setminus A$ is open. Give $x\in X\setminus A$ there must be some $\varepsilon>0$ such that $B_\varepsilon^{(X,\,d)}(x)\cap A=\emptyset$. Otherwise, if for all $\varepsilon>0$ there is an $n\in\mathbb{N}$ such that $d(x,\,a_n)<\varepsilon$ we could find a subsequence of a that converges to x, contradicting the claim that a has no such subsequences. So about every element of $X\setminus A$ we can place an ε ball contained entirely inside of $X\setminus A$, and hence this set is open. Since $X\setminus A$ is open, A is closed.

Moreover, $A \setminus \{ a_0 \}$ is closed. Since no subsequence of a converges to a_0 there must be some $\varepsilon > 0$ such that $B_{\varepsilon}^{(X,\, d)}(a_0) \cap A = \{ a_0 \}$. Since open balls are open, the set $A \setminus B_{\varepsilon}^{(X,\, d)}(a_0)$ is the difference of an open set from a closed set, which is closed. Since $B_{\varepsilon}^{(X,\, d)}(a_0) \cap A = \{ a_0 \}$ we have that $A \setminus B_{\varepsilon}^{(X,\, d)}(a_0) = A \setminus \{ a_0 \}$. So $A \setminus \{ a_0 \}$ is closed. Even more, for all $n \in \mathbb{N}$ if we define the set:

$$B_n = \{ a_0,\, \dots,\, a_{n-1} \} \tag{2}$$

then the set $A \setminus B_n$ is closed. Denote this set by $\mathcal{C}_n$:

$$\mathcal{C}_n = A \setminus B_n \tag{3}$$

But then $\mathcal{C} : \mathbb{N} \to \mathcal{P}(X)$ is a nested sequence of non-empty closed sets, so the intersection is non-empty by hypothesis. But:

$$\begin{aligned} \bigcap_{n \in \mathbb{N}} \mathcal{C}_n &= \bigcap_{n \in \mathbb{N}} (A \setminus B_n) && (4) \\ &= A \setminus \bigcup_{n \in \mathbb{N}} B_n && (5) \\ &= A \setminus A && (6) \\ &= \emptyset && (7) \end{aligned}$$

A contradiction. Hence $(X,\, d)$ is compact.

As mentioned there are many ways to prove this claim. Let's use open sets. We proved that a metric space is compact if and only if for every open cover $\mathcal{O}$ of X there is a finite subcover $\Delta \subseteq \mathcal{O}$. Suppose $(X,\, d)$ is compact and let $\mathcal{C} : \mathbb{N} \to \mathcal{P}(X)$ be a nested sequence of closed non-empty sets with the property that $\bigcap_{n \in \mathbb{N}} \mathcal{C}_n = \emptyset$. Define:

$$\mathcal{U}_n = X \setminus \mathcal{C}_n \tag{8}$$

Then the set $\mathcal{O} = \{ \mathcal{U}_n \mid n \in \mathbb{N} \}$ is and open cover of X since:

$$\begin{aligned} X &= X \setminus \emptyset && (9) \\ &= X \setminus \bigcap_{n \in \mathbb{N}} \mathcal{C}_n && (10) \\ &= \bigcup_{n \in \mathbb{N}} (X \setminus \mathcal{C}_n) && (11) \\ &= \bigcup_{n \in \mathbb{N}} \mathcal{U}_n && (12) \\ &= \bigcup \mathcal{O} && (13) \end{aligned}$$

But $(X,\, d)$ is compact, so there is a finite subcover $\Delta = \{ \mathcal{U}_{n_0},\, \dots,\, \mathcal{U}_{n_m} \}$. Let $\mathcal{U}_N$ be the element of Δ with the largest index. Since the sets $\mathcal{C}_n$ are nested, $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$, so are the sets $\mathcal{U}_n$. That is, $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$. But then $\{ \mathcal{U}_N \}$ is an open cover of X, meaning $\mathcal{U}_N = X$. But then $\mathcal{C}_N = X \setminus \mathcal{U}_N$ is empty, contradicting the fact that all $\mathcal{C}_n$ are non-empty.

To use open sets for the converse of this statement we need the fact that a metric space is compact if and only if it is *countably* compact. That is, for every *countable* open cover $\mathcal{O}$ of X there is a finite subcover $\Delta \subseteq \mathcal{O}$ of X. Compactness certainly implies countable compactness, since we can consider arbitrary open covers $\mathcal{O}$, not just countable ones. Let's prove that countably compact metric spaces are compact. If not, then there is a sequence $a : \mathbb{N} \to X$ with no convergent subsequence. The set A described previously is closed:

$$A = \{a_n \in X \mid n \in \mathbb{N}\} \tag{14}$$

and the sets $\mathcal{C}_n = A \setminus B_n$ are also closed. Moreover they are nested, $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$, and hence the sets:

$$\mathcal{U}_n = X \setminus \mathcal{C}_n \tag{15}$$

are open and nested, $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$. By similar reasoning as before, the collection $\mathcal{O} = \{\mathcal{U}_n \mid n \in \mathbb{N}\}$ forms an open cover of X. More than that, it is a *countable* open cover, and since $(X,\, d)$ is countably compact there is a finite subcover $\Delta \subseteq \mathcal{O}$. Since the sets are nested, we may choose $\Delta = \{\mathcal{U}_N\}$ for some $N \in \mathbb{N}$. But then $\mathcal{U}_N = X$, meaning $\mathcal{C}_N = \emptyset$. This implies that the set A is finite. But a sequence $a : \mathbb{N} \to A$ into a finite set must have a convergent subsequence, a contradiction. Hence $(X,\, d)$ is compact.

Note: Similar arguments do not hold for topological spaces. Countable compactness and compactness can be different.

Using this, let's show that a metric space with the nested intersection property is countably compact (and hence compact). Suppose not. Then there is a countable open cover $\mathcal{O}$ of $(X,\, d)$ with no finite subcover. Since it is countable there is a surjection $\mathcal{U} : \mathbb{N} \to \mathcal{O}$. That is, we may list the elements as:

$$\mathcal{O} = \{\mathcal{U}_0, \mathcal{U}_1, \ldots, \} \tag{16}$$

Define $\mathcal{V} : \mathbb{N} \to \tau_d$ as follows:

$$\mathcal{V}_N = \bigcup_{n=0}^{N} \mathcal{U}_n \tag{17}$$

The sets $\mathcal{V}_n$ are open, being the union of open sets, and nested, $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$. Moreover, since $\mathcal{O}$ has no finite subcover, $\mathcal{V}_n \neq X$ for all $n \in \mathbb{N}$. But then $\mathcal{C}_n = X \setminus \mathcal{V}_n$ is a sequence of non-empty nested closed sets. But then, by hypothesis, $\bigcap_{n\in\mathbb{N}} \mathcal{C}_n$ is non-empty. Let $x \in \bigcap_{n\in\mathbb{N}} \mathcal{C}_n$. Then $x \in \mathcal{C}_n$ for all $n \in \mathbb{N}$, and hence $x \notin \mathcal{V}_n$ for all $n \in \mathbb{N}$. But $\mathcal{O}$ is an open cover, so $x \in \mathcal{U}_n$ for some $n \in \mathbb{N}$, which implies $x \in \mathcal{V}_n$, a contradiction. Hence $(X,\, d)$ is countably compact. Since countably compact metric spaces are compact, $(X,\, d)$ is also compact.

The second part of the problem comes straight from the first part. If every sequence $\mathcal{U} : \mathbb{N} \to \tau_d$ of nested open proper subsets of X is such that $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n \neq X$, then by taking complements we see that every sequence of nested non-empty closed sets $\mathcal{C} : \mathbb{N} \to \mathcal{P}(X)$ has non-empty intersection, and therefore the space is compact. Similarly, if the space is compact, then we have the nested intersection property for closed sets. By looking at the complement we see that the union of nested open proper subsets cannot be the entire space since it cannot contain the element common to the intersection of the closed sets. So this is yet another equivalent definition of compactness in a metric space. $\square$

Problem 4 (Calculus)

With our tools from metric space theory, one of the harder theorems from calculus becomes quite simple.

- (4 Points) Prove that if (X, d_X) is compact, if (Y, d_Y) is a metric space, and if $f : X \to Y$ is continuous, then $f[X] \subseteq Y$ is a compact subspace.
- (4 Points) The extreme value theorem states that if $f : [a, b] \to \mathbb{R}$ is continuous, then there is $c_{\text{min}}, c_{\text{max}} \in [a, b]$ such that $f(c_{\text{min}}) \leq f(x) \leq f(c_{\text{max}})$ for all $x \in [a, b]$. Let's take that up a notch. Prove that if (X, d) is compact, and if $f : X \to \mathbb{R}$ is continuous, then there are points c_{min} and c_{max} such that $f(c_{\text{min}}) \leq f(x) \leq f(c_{\text{max}})$ for all $x \in X$. [Hint: The previous part is enormously helpful.]

Solution. Let $b : \mathbb{N} \to f[X]$ be a sequence. Since $f : X \to f[X]$ is surjective there is a right-inverse $g : f[X] \to X$ with the property that $(f \circ g)(x) = x$. Let $a_n = g(b_n)$. Since (X, d) is compact there is a convergent subsequence a_k. Let x be the limit, $a_{k_n} \to x$. But f is continuous, so $f(a_{k_n}) \to f(x)$. But $f(a_{k_n}) = b_{k_n}$, so $b_{k_n} \to f(x)$, and hence b has a convergent subsequence. Thus, $f[X]$ is a compact subspace of (Y, d_Y).

To prove the extreme value theorem, note that $f[X] \subseteq \mathbb{R}$ is compact, so by Heine-Borel it is closed and bounded. Since it is bounded there is a least upper bound y_{max} and a greatest lower bound y_{min}. Since $f[X]$ is closed y_{min} and y_{max} are elements for $f[X]$. That is, there are $x_{\text{min}}, x_{\text{max}} \in X$ such that $f(x_{\text{min}}) = y_{\text{min}}$ and $f(x_{\text{max}}) = y_{\text{max}}$. But then, since y_{min} and y_{max} are the greatest lower bound and least upper bound of $f[X]$, respectively, for all $x \in X$ we have $f(x_{\text{min}}) \leq f(x)$ and $f(x) \leq f(x_{\text{max}})$, which is the desired property. □

Problem 5 (Product Spaces)

(6 Points) Given metric spaces $(X,\, d_X)$ and $(Y,\, d_Y)$, prove that all three product metrics are topologically equivalent:

$$d_1\big((x_0,\, y_0),\, (x_1,\, y_1)\big) = d_X(x_0,\, x_1) + d_Y(y_0,\, y_1) \tag{18}$$

$$d_2\big((x_0,\, y_0),\, (x_1,\, y_1)\big) = \sqrt{d_X(x_0,\, x_1)^2 + d_Y(y_0,\, y_1)^2} \tag{19}$$

$$d_\infty\big((x_0,\, y_0),\, (x_1,\, y_1)\big) = \max\big(d_X(x_0,\, x_1),\, d_Y(y_0,\, y_1)\big) \tag{20}$$

Solution. Topological equivalence is an equivalence relation, essentially since equality is. If d_0 is topologically equivalent to d_1, then $\tau_{d_0} = \tau_{d_1}$, and hence $\tau_{d_1} = \tau_{d_0}$, so d_1 is topologically equivalent to d_0. Reflexivity and transitivity can similarly be checked. So let's prove the metrics d_1 and d_∞ are equivalent, as are the metrics d_2 and d_∞. Let's start with d_1 and d_∞. Let $r > 0$. We must find $r' > 0$ and $r'' > 0$ such that:

$$B_{r'}^{(X,\, d_1)}\big((x,\, y)\big) \subseteq B_r^{(X,\, d_\infty)}\big((x,\, y)\big) \tag{21a}$$

$$B_{r''}^{(X,\, d_\infty)}\big((x,\, y)\big) \subseteq B_r^{(X,\, d_1)}\big((x,\, y)\big) \tag{21b}$$

Let $r' = r$ and $r'' = r/2$. Then:

$$d_1\big((x_0,\, y_0),\, (x_1,\, y_1)\big) < r' \tag{22a}$$

$$\Rightarrow d_X(x_0,\, x_1) + d_Y(y_0,\, y_1) < r' \tag{22b}$$

$$\Rightarrow \max\big(d_X(x_0,\, x_1),\, d_Y(y_0,\, y_1)\big) < r' \tag{22c}$$

$$\Rightarrow \max\big(d_X(x_0,\, x_1),\, d_Y(y_0,\, y_1)\big) < r \tag{22d}$$

And therefore:

$$B_{r'}^{(X,\, d_1)}\big((x,\, y)\big) \subseteq B_r^{(X,\, d_\infty)}\big((x,\, y)\big) \tag{23}$$

In the other direction:

$$\max\big(d_X(x_0,\, x_1),\, d_Y(y_0,\, y_1)\big) < r'' \tag{24a}$$

$$\Rightarrow 2\max\big(d_X(x_0,\, x_1),\, d_Y(y_0,\, y_1)\big) < 2r'' \tag{24b}$$

$$\Rightarrow d_X(x_0,\, x_1) + d_Y(y_0,\, y_1) < 2r'' \tag{24c}$$

$$\Rightarrow d_1\big((x_0,\, y_0),\, (x_1,\, y_1)\big) < r \tag{24d}$$

so we may conclude:

$$B_{r''}^{(X,\, d_\infty)}\big((x,\, y)\big) \subseteq B_r^{(X,\, d_1)}\big((x,\, y)\big) \tag{25}$$

Now to compare d_2 and d_∞. Again, choose $r' = r$. We get:

$$d_2\big((x_0,\, y_0),\, (x_1,\, y_1)\big) < r' \tag{26a}$$

$$\Rightarrow \sqrt{d_X(x_0,\, x_1)^2 + d_Y(y_0,\, y_1)^2} < r' \tag{26b}$$

$$\Rightarrow \max\big(d_X(x_0,\, x_1),\, d_Y(y_0,\, y_1)\big) < r' \tag{26c}$$

$$\Rightarrow \max\big(d_X(x_0,\, x_1),\, d_Y(y_0,\, y_1)\big) < r \tag{26d}$$

and so:

$$B_{r'}^{(X,\,d_2)}\big((x,\,y)\big) \subseteq B_r^{(X,\,d_\infty)}\big((x,\,y)\big) \tag{27}$$

choosing $r'' = r/\sqrt{2}$ we get:

$$\max\big(d_X(x_0,\,x_1),\,d_Y(y_0,\,y_1)\big) < r'' \tag{28a}$$

$$\Rightarrow \max\big(d_X(x_0,\,x_1),\,d_Y(y_0,\,y_1)\big)\sqrt{2} < r''\sqrt{2} \tag{28b}$$

$$\Rightarrow \sqrt{2\max\big(d_X(x_0,\,x_1),\,d_Y(y_0,\,y_1)\big)^2} < r''\sqrt{2} \tag{28c}$$

$$\Rightarrow \sqrt{d_X(x_0,\,x_1)^2 + d_Y(y_0,\,y_1)^2} < r''\sqrt{2} \tag{28d}$$

$$\Rightarrow d_2\big((x_0,\,y_0),\,(x_1,\,y_1)\big) < r \tag{28e}$$

Hence:

$$B_{r''}^{(X,\,d_\infty)}\big((x,\,y)\big) \subseteq B_r^{(X,\,d_2)}\big((x,\,y)\big) \tag{29}$$

so d_2 and d_∞ are topologically equivalent. □