Point-Set Topology: Homework 3

Summer 2023

Problem 1 (Separability)

A separable topological space is a space (X, τ) such that there is a countable subset $A \subseteq X$ such that $\mathrm{Cl}_\tau(A) = X$. A metric space is separable if and only if it is second-countable. This feature is special to metric spaces. Take $\mathbb{R}$ with the standard topology, and equip $\mathbb{R}/\mathbb{Z}$ with the quotient topology. Intuitively this is infinite many circles all touching at 0. It is not first-countable, and hence not second-countable, even though $\mathbb{R}$ is. It is still separable.

- (6 Points) Let (X, τ) be a separable topological space. Let R be any equivalence relation on X. Prove that $(X/R, \tau_{X/R})$ is separable. That is, separability is a topological property preserved by quotients.

Solution. Since (X, τ) is separable, there is a countable dense subset A. Let $q : X \rightarrow X/R$ be the quotient map, $q(x) = [x]$ where $[x]$ is the equivalence class of x, and let $B = q[A]$. Since B is the image of a countable set, it too is countable. We must prove $\mathrm{Cl}_{\tau_{X/R}}(B) = X/R$. Let $\tilde{\mathcal{C}} \subseteq X/R$ be a closed set containing B. But q is continuous, and so the pre-image of closed sets is closed, meaning $q^{-1}[\tilde{\mathcal{C}}]$ is closed. Let $\mathcal{C} = q^{-1}[\tilde{\mathcal{C}}]$. Then $\mathcal{C}$ is a closed set that contains A since $q[A] = B$. But if $A \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed, then $\mathrm{Cl}_\tau(A) \subseteq \mathcal{C}$. But $\mathrm{Cl}_\tau(A) = X$, and hence $\mathcal{C} = X$. But then, since quotient maps are surjective, we have that $q[\mathcal{C}] = q[X] = X/R$. But $q[\mathcal{C}] \subseteq \tilde{\mathcal{C}}$, by definition of $\mathcal{C}$ and $\tilde{\mathcal{C}}$, and hence $\tilde{\mathcal{C}} = X/R$. That is, if $\tilde{\mathcal{C}}$ is a closed subset of X/R such that $B \subseteq \tilde{\mathcal{C}}$, then $\tilde{\mathcal{C}} = X/R$. Hence $\mathrm{Cl}_{\tau_{X/R}}(B) = X/R$, so B is a countable dense subset and $(X/R, \tau_{X/R})$ is separable. □

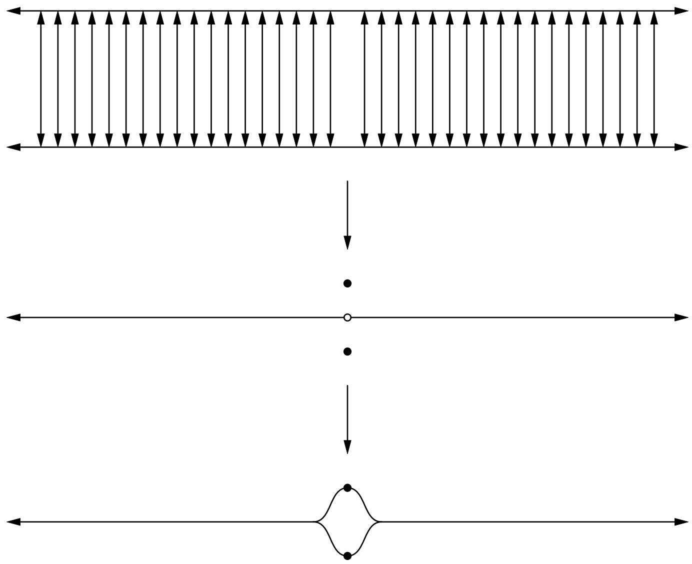

Figure 1: The Bug-Eyed Line Construction

Problem 2 (Embeddings)

The bug-eyed line is a quotient space of $X = \mathbb{R} \times \{0, 1\}$ where $\mathbb{R}$ has the standard Euclidean topology and $\{0, 1\}$ has the discrete topology. X is given the product topology. We identity $(x, 0)$ with $(x, 1)$ for all $x \neq 0$ and then take the quotient of X under this relation. This idea is shown in Fig. 1

- (6 Points) Prove that it is impossible to embed the bug-eyed line into $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

Solution. This space is not Hausdorff. For let $q : X \to X/R$ be the canonical quotient map, and define $0' = q((0, 0))$ and $0'' = q((0, 1))$. These are the two origins in the buy-eyed line. Let $\mathcal{U} \subseteq X/R$ be an open set about $0'$ and $\mathcal{V} \subseteq X/R$ be an open set about $0''$. Then $q^{-1}[\mathcal{U}], q^{-1}[\mathcal{V}] \subseteq X$ are open subsets of X since q is continuous. Moreover, $(0, 0)$ is an element of $q^{-1}[\mathcal{U}]$ and $(0, 1)$ is an element of $q^{-1}[\mathcal{V}]$. But the topology on X is the product topology from $\mathbb{R}$ and $\mathbb{Z}_2 = \{0, 1\}$, the latter given the discrete topology. So an open subset about $(0, 0)$ must contain all points between $(-\varepsilon, 0)$ and $(\varepsilon, 0)$. A similar statement can be made for $(0, 1)$. Since $(x, 0)$ and $(x, 1)$ are identified by the relation for all $x \neq 0$, projecting these open intervals down to the quotient space shows that $\mathcal{U}$ and $\mathcal{V}$ must overlap. Hence $(X/R, \tau_{X/R})$ is not Hausdorff. But if $f : X/R \to \mathbb{R}^n$ is an embedding, then X/R is homeomorphic to the subspace $f[X/R] \subseteq \mathbb{R}^n$. But $\mathbb{R}^n$ is Hausdorff, so all of its subspaces are Hausdorff. So no such embedding could possibly exist.

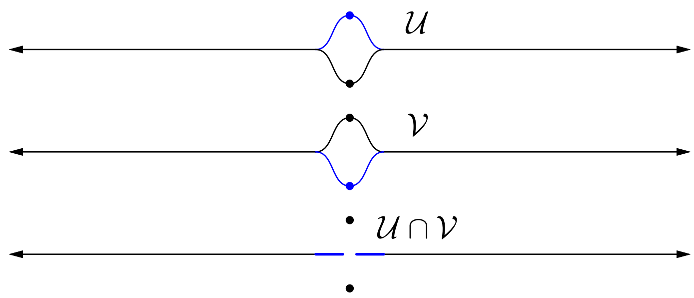

Figure 2: Open Sets in the Bug-Eyed Line

□

Fig. 2 provides a visual of the description of open sets given in solution. Any open set containing the first origin must overlap with any open set containing the second. We imagine the bug-eyed line as the real line with an extra origin that is almost indistinguishable from the first. This space is Fréchet, however. The two origins are indeed still closed.

Problem 3 (Quotients)

Let $X = \mathbb{R}/\mathbb{Q}$, equipped with the quotient topology where $\mathbb{R}$ carries the usual Euclidean topology.

- (4 Points) Is this space Hausdorff? Is it Fréchet?

Solution. This space is not Hausdorff. A subset of $\mathbb{R}/\mathbb{Q}$ is open if and only if the pre-image is open. This is one of the defining characteristics of the quotient map q. Let $[x], [y] \in \mathbb{R}/\mathbb{Q}$. There are three possibilities. Both x and y are irrational, only one of x and y are irrational, and both x and y are rational. If x and y are rational, then $[x] = [y]$, so we may discard this possibility. Suppose both x and y are irrational. Let $\mathcal{U}, \mathcal{V}$ be open sets about $[x]$ and $[y]$, respectively. Then $q^{-1}[\mathcal{U}]$ and $q^{-1}[\mathcal{V}]$ are open sets containing x and y, respectively. But open sets in $\mathbb{R}$ are described by the metric. So there is some $\varepsilon_x > 0$ and some $\varepsilon_y > 0$ such that $(x - \varepsilon_x,\, x + \varepsilon_x) \subseteq q^{-1}[\mathcal{U}]$ and $(y - \varepsilon_y,\, y + \varepsilon_y) \subseteq q^{-1}[\mathcal{V}]$. But there must be a rational number between $x + \varepsilon_x$ and a rational number between $y + \varepsilon_y$. But all rationals are identified together by the equivalence relation, meaning these rationals map to the same point under q. Hence $\mathcal{U}$ and $\mathcal{V}$ must overlap, so $[x]$ and $[y]$ can not be separated by open sets.

If x is irrational and y is rational, the argument is almost identical. Any open set about $[x]$ must contain $[y]$ by the previous argument, and hence $[x]$ and $[y]$ can not be separated by open sets either. So $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

The space is also not Fréchet. Given irrational numbers x and y we can indeed find open sets for $[x]$ and $[y]$ satisfying the Fréchet condition. Namely, set $\mathcal{U} = \mathbb{R} \setminus \{\, y \,\}$ and $\mathcal{V} = \mathbb{R} \setminus \{\, x \,\}$. These sets are open in $\mathbb{R}$ and moreover they are saturated with respect to q. Hence $q[\mathcal{U}]$ and $q[\mathcal{V}]$ are open sets with $[x] \in q[\mathcal{U}]$, $[x] \notin q[\mathcal{V}]$, and $[y] \in q[\mathcal{V}]$ and $[y] \notin q[\mathcal{U}]$. The problem arises when we consider x irrational and y rational. When exploring the Hausdorff property we saw that any open set containing $[x]$ must also contain $[y]$, and hence the Fréchet condition cannot be satisfied. $\mathbb{R}/\mathbb{Q}$ is neither Hausdorff nor Fréchet. $\square$

Problem 4 (Products)

Consider topological spaces $(X,\,\tau_X)$, $(Y,\,\tau_Y)$, and $(Z,\,\tau_Z)$. Equip $X\times Y$ with the product topology $\tau_{X\times Y}$.

- (6 Points) Prove that a function $f:Z\rightarrow X\times Y$ is continuous if and only if the component functions $\mathrm{proj}_X\circ f:Z\rightarrow X$ and $\mathrm{proj}_Y\circ f:Z\rightarrow Y$ are continuous.

Solution. One direction is easier than the other. If f is continuous, then since projections are continuous, $\mathrm{proj}_X\circ f$ and $\mathrm{proj}_Y\circ f$ are the compositions of continuous functions, which are therefore continuous. That is, if f is continuous, then so are the component functions. In the other direction, suppose the component functions are continuous. Let $z\in Z$ and $\mathcal{W}\in\tau_{X\times Y}$ be any open set containing $f(z)$. Since $\tau_{X\times Y}$ has as a basis the set of all open rectangles, there must be some open sets $\mathcal{U}\in\tau_X$ and $\mathcal{V}\in\tau_Y$ such that $f(z)\in\mathcal{U}\times\mathcal{V}$ and $\mathcal{U}\times\mathcal{V}\subseteq\mathcal{W}$. But $\mathrm{proj}_X\circ f$ is continuous, and $(\mathrm{proj}_X\circ f)(z)\in\mathcal{U}$, so there is an open set $\mathcal{E}_X$ such that $z\in\mathcal{E}_X$ and $(\mathrm{proj}_X\circ f)[\mathcal{E}_X]\subseteq\mathcal{U}$. Similarly there exists a set $\mathcal{E}_Y$ for $\mathrm{proj}_Y\circ f$. Let $\mathcal{E}=\mathcal{E}_X\cap\mathcal{E}_Y$. Then $\mathcal{E}$ is open, being the intersection of two open sets, and $z\in\mathcal{E}$. Moreover, $f[\mathcal{E}]\subseteq\mathcal{U}\times\mathcal{V}$. For let $(x,\,y)\in f[\mathcal{E}]$. Then $\mathrm{proj}_X\big((x,\,y)\big)=x$ and hence $x\in\mathcal{U}$, and $\mathrm{proj}_Y\big((x,\,y)\big)=y$ and so $y\in\mathcal{V}$. But then $(x,\,y)\in\mathcal{U}\times\mathcal{V}$, meaning $f[\mathcal{E}]\subseteq\mathcal{U}\times\mathcal{V}$. But $\mathcal{U}\times\mathcal{V}\subseteq\mathcal{W}$, and hence $f[\mathcal{E}]\subseteq\mathcal{W}$. That is, for all $z\in Z$ and for every open set $\mathcal{W}$ containing $f(z)$ there is an open set $\mathcal{E}\subseteq Z$ such that $z\in\mathcal{E}$ and $f[\mathcal{E}]\subseteq\mathcal{W}$. So f is continuous. $\square$