Point-Set Topology: Lecture 1

Ryan Maguire

October 6, 2023

Contents

1 Motivation

Topology is the study of *continuous deformations*. Continuously changing a space without tearing or cutting. Ask yourself if you think it is possible to continuously morph a square into a circle. See Fig. 1 for a demonstration. Now ask yourself if it is possible to change a sphere into a torus (Fig. 2). The first problems in topology stem from graph theory, such as the bridges of Königsberg problem. You are asked to traverse the seven bridges in the city of Königsberg, crossing each bridge exactly once (Fig. 3). Is this possible?

We can simplify the problem by treating plots of land as single dots, and bridges as curves and line segments, resulting in the graph shown in Fig. 4. The problem asks you to traverse each edge of this graph once and only once. This turns out to be impossible. Turning this into a graph problem is the start of topology.

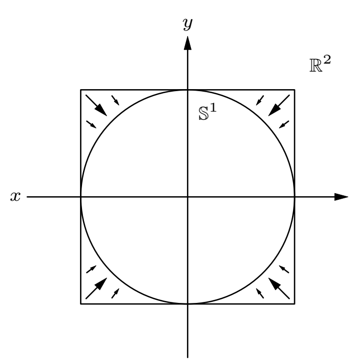


Figure 1: Changing a Square to a Circle

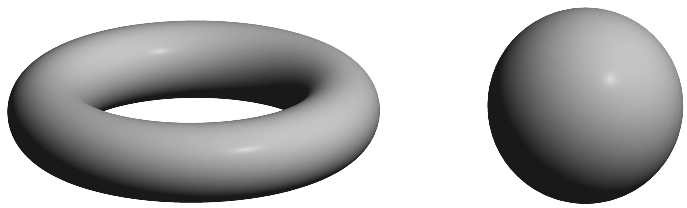


Figure 2: A Torus and a Sphere

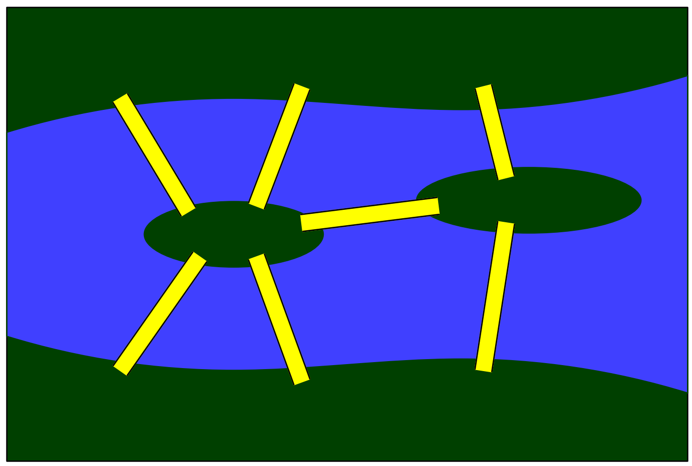


Figure 3: The Seven Bridges of Königsberg

We may move the edges and vertices of the graph around without altering the question. If we cut an edge or remove a vertex, we will change the problem. That is, we may change the graph *continuously* without affecting anything.

The next discovery that motivates topology is the *Euler characteristic*. A polyhedron is a collection of vertices, edges, and faces. A convex polyhedron has the property that if it has V vertices, E edges, and F faces, then:

$$V - E + F = 2 \tag{1}$$

The convexity is not actually needed. The formula $V - E + F = 2$ is true as long as the polyhedron can be *continuously deformed* into a sphere. If we have a polyhedron that can be continuously deformed into a torus, we get a different number. Examining Fig. 5.2 we count:

$$V - E + F = 0 \tag{2}$$

This *alternating sum* is called the *Euler characteristic* of the polyhedron and it is preserved by continuous deformations.

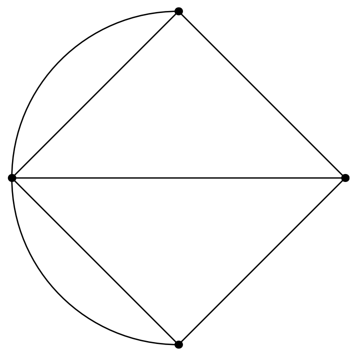
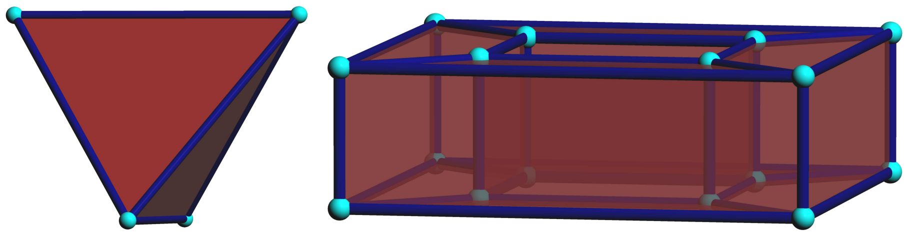


Figure 4: The Graph of the Seven Bridges of Königsberg



5.1: A Tetrahedron

5.2: A Polyhedral Torus

Figure 5: Examples of Polyhedra

2 Sets

Sets are the main object dealt with in most branches of mathematics.

Definition 2.1 (**Set**) A set is a collection of objects called elements. ■

Sets are often denoted using capital Latin letters such as A, B, C, X, Y, and Z. Elements of sets are usually denoted by lower case Latin letters like a, b, c, x, y, and z. This need not always be the case, but is fairly common.

Example 2.1 Finite sets are denoted with braces and commas separating the elements. For example, if A is the set consisting of the elements 1, 2, and 3, we write:

$$A = \{\,1,\,2,\,3\,\} \tag{3}$$

Order does not matter for sets, we can also write:

$$A = \{\,3,\,1,\,2\,\} \tag{4}$$

And repetition is also irrelevant:

$$A = \{\,1,\,1,\,2,\,3\,\} \tag{5}$$

All of these mean the set A. ■

Notation 2.1 (**Containment**) If A is a set and a is an element of A, we write $a \in A$. If a is not an element of A, we write $a \notin A$. ■

The expression $a \in A$ reads aloud as *a is in A* or *a is an element of A*. Similarly, $a \notin A$ reads aloud as *a is not in A*.

Example 2.2 Using $A = \{1, 2, 3\}$, we have that $1 \in A$, $2 \in A$, and $3 \in A$ since 1, 2, and 3 are elements of the set A. However, $4 \notin A$ since 4 is not an element of A. ■

Sets are usually constructed with set-builder notation. If we have some set X that we know exists, we can take a sentence P and apply it to all elements $x \in X$. This is expressed

$$A = \{\, x \in X \mid P(x) \,\} \tag{6}$$

which reads *A is the set of all x in X such that $P(x)$ is true.*

Example 2.3 Let $\mathbb{N}$ be the set of natural numbers

$$\mathbb{N} = \{\, 0, 1, 2, 3, 4, \dots \} \tag{7}$$

Let $P(n)$ be the sentence *n is even.* Consider the set

$$A = \{\, n \in \mathbb{N} \mid P(n) \,\} \tag{8}$$

This is the set of all natural numbers that are even. We can write this explicitly as follows:

$$A = \{\, 0, 2, 4, \dots \} \tag{9}$$

■

Be careful of abusing set-builder notation. If you have some sentence $P(x)$ you can **not** write the following:

$$A = \{\, x \mid P(x) \,\} \tag{10}$$

That is, you may not consider the set of all x such that $P(x)$ is true. You may only apply your sentence to some set X you already know exists, and then collect all $x \in X$ satisfying your sentence P. Failure to do this results in Russell's paradox. Consider the sentence $P(x) =$ *x is a set.* Then $A = \{\, x \mid P(x) \,\}$ is the *set of all sets*. Is A an element of itself? If allowed, it is possible to construct a statement that is both true and false, something that should be avoided.

Definition 2.2 (**Subset**) A subset of a set B is a set A such that for all $a \in A$ it is true that $a \in B$. We write $A \subseteq B$. ■

Subsets can be described by blobs in the plane. See Fig. 6.

The axioms of set theory provide the existence of four operations on sets: union, intersection, set difference, and symmetric difference.

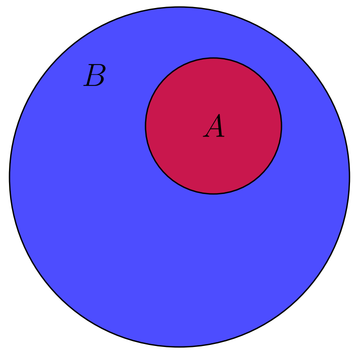
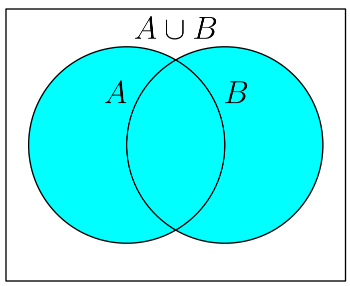
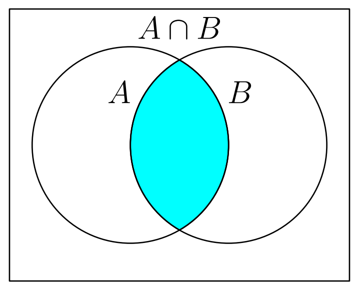


Figure 6: Diagram for Subsets



7.1: Venn Diagram for Union



7.2: Venn Diagram for Intersection

Definition 2.3 (**Union of Two Sets**) The union of two sets A and B is the set $A \cup B$ defined by:

$$A \cup B = \{\, x \mid x \in A \text{ or } x \in B \,\} \tag{11}$$

That is, the set of all elements in A, or in B, or in both. ∎

The union of two sets can be visualized with a *Venn diagram* (Fig. 7.1). It is worth detailing what the word *or* means. In English there are two uses of the word or, the *inclusive or* and the *exclusive or*. Given two sentences P and Q, the exclusive or means P is true or Q is true, but not both. The inclusive or means P is true, Q is true, or both are true. In mathematics we always use the inclusive or. The exclusive or is denoted *xor*. It is a useful concept in computer science and analysis of algorithms, but we will not need it in topology.

Example 2.4 Let $A = \{\, 1,\ 2,\ 3 \,\}$ and $B = \{\, 3,\ 4,\ 5 \,\}$. The union is then:

$$A \cup B = \{\, 1,\ 2,\ 3,\ 4,\ 5 \,\} \tag{12}$$

Remember, sets have no notion of repetition, so including 3 twice would be redundant. ∎

Definition 2.4 (**Intersection of Two Sets**) The intersection of two sets A and B is the set $A \cap B$ defined by:

$$A \cap B = \{\, x \mid x \in A \text{ and } x \in B \,\} \tag{13}$$

That is, the set of all elements that are in both A and B simultaneously. ∎

The word *and* is easier to understand since its mathematical use matches the grammatical one. Given two sentences P and Q, the statement P *and* Q means both P is true and Q is true, and both are true simultaneously.

Example 2.5 Let $A = \{\, 1, 2, 3 \,\}$ and $B = \{\, 3, 4, 5 \,\}$. The intersection is then:

$$A \cap B = \{\, 3 \,\} \tag{14}$$

The only element A and B have in common is 3. ∎

Intersection can also be represented by a Venn diagram (See Fig. 7.2). What if A and B have no sets in common? Let's consider the following example.

Example 2.6 Let $A = \{\, 1, 2, 3 \,\}$ and $B = \{\, 4, 5, 6 \,\}$. The intersection $A \cap B$ is the set of elements in both A and B. But A and B have no elements in common, so the intersection is *empty*. This is the empty set, denoted $\emptyset$, and occasionally written as $\emptyset = \{\,\}$. We have $A \cap B = \emptyset$. ∎

The axioms of set theory provide the existence of the empty set, but we'll just take it as a definition and move on.

Definition 2.5 (**The Empty Set**) The empty set is the unique set $\emptyset$ that contains no elements. That is, for all x, it is true that $x \notin \emptyset$. We may write $\emptyset = \{\,\}$ for convenience. ∎

Definition 2.6 (**Disjoint Sets**) Disjoint sets are sets A and B that have no elements in common. That is, sets A and B such that $A \cap B = \emptyset$. ∎

The sets in Ex. 2.6 are disjoint.

The next operation is set difference. It is somewhat like subtraction for sets.

Definition 2.7 (**Set Difference**) The set difference of a set A from a set B is the set $B \setminus A$ defined by:

$$B \setminus A = \{\, x \in B \mid x \notin A \,\} \tag{15}$$

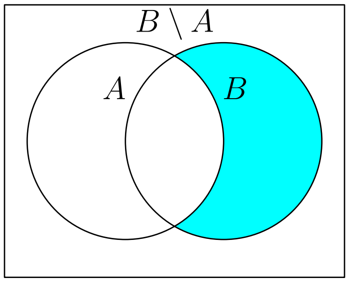
That is, the set of all elements in B that are not in A. ∎

The Venn diagram for set difference is given in Fig. 8.1.

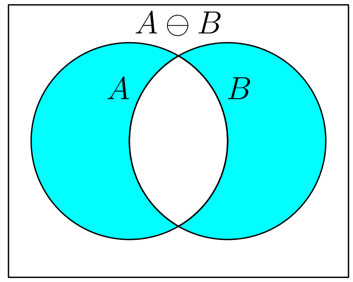
Example 2.7 Let $A = \{\, 1, 2, 3 \,\}$ and $B = \{\, 3, 4, 5 \,\}$. The set difference $B \setminus A$ is:

$$B \setminus A = \{\, 4, 5 \,\} \tag{16}$$

That is, the only element common to A and B is 3, so $B \setminus A$ is everything in B except for 3. ∎



8.1: Venn Diagram for Set Difference



8.2: Venn Diagram for Symmetric Difference

The last operation is *symmetric difference* and it is defined in terms of the other three. We will not use it, but for completeness the Venn diagram is shown.

Definition 2.8 (**Symmetric Difference**) The symmetric difference of two sets A and B is the set $A \ominus B$ defined by:

$$A \ominus B = (A \cup B) \setminus (A \cap B) \tag{17}$$

That is, the set of all elements in either A or B, but not both. ■

Symmetric difference is the set equivalent of the exclusive or. The Venn diagram is shown in Fig. 8.2.

There are several theorems relating union, intersection, and set difference. We will use them often.

$$A \cup B = B \cup A \quad \text{(Commutativity of Unions)}$$
$$A \cap B = B \cap A \quad \text{(Commutativity of Intersections)}$$
$$A \cup (B \cup C) = (A \cup B) \cup C \quad \text{(Associativity of Unions)}$$
$$A \cap (B \cap C) = (A \cap B) \cap C \quad \text{(Associativity of Intersections)}$$
$$A \cup \emptyset = A \quad \text{(Identity Law of Unions)}$$
$$A \subseteq B \Rightarrow A \cap B = A \quad \text{(Identity Law of Intersections)}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{(Distributive Law of Unions)}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{(Distributive Law of Intersections)}$$
$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \quad \text{(De Morgan's Law of Unions)}$$
$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) \quad \text{(De Morgan's Law of Intersections)}$$

Lastly, a brief discussion on *implication*. This is the use of the expression *if-then* in mathematics. Consider the sentence

if I am late to work, then I will be fired

This contains two sentences. $P(x)$ is *x is late for work* and $Q(x)$ is *x will be fired*. There are four possibilities for the sentence *if P, then Q*. Let's work through them.

P	Q	$P \Rightarrow Q$
False	False	True
False	True	True
True	False	False
True	True	True

Table 1: Truth Table for Implication

I was not late to work, and I was not fired. Is the statement *if I am late to work, then I will be fired* true or false? The situation that triggers the firing didn't happen so we certainly cannot conclude that the statement is false. We thus say that in this scenario the sentence is true.

I was not late to work, and I was fired. Is the statement false? Of course not, there are plenty of reasons for being fired. Perhaps I have a sailor's tongue or sloppy handwriting. In this situation we conclude that the sentence is true.

I was late to work, and I was not fired. In this event, my boss is very nice, however the sentence is false. I was late to work and yet I was not fired.

I was late to work, and I was fired. This is perhaps the easiest to grasp since it's the scenario one intuitively thinks about when hearing *if-then* sentences. In this case the sentence is true.

The technical name for *if-then* is *implication*. The symbol for this is a large arrow $\Rightarrow$. The truth table for implication is given in Tab. 1.