Point-Set Topology: Lecture 2

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1 Cartesian Product and Power Sets

The axioms of set theory allow for two more constructions from sets. If A is a set, we can think of the set of all *subsets* of A.

Definition 1.1 (Power Set) The power set of a set A is the set $\mathcal{P}(A)$ defined by:

$$\mathcal{P}(A) = \{\, B \mid B \subseteq A \,\} \tag{1}$$

That is, the set of all subsets of A. ∎

Note that the set-builder notation has been abused here. I did not write the power set equation in the form $Y = \{\, x \in X \mid P(x) \,\}$ where X is a set *that we know exists* and P is some sentence on that set. Instead, I did the exact thing that can lead to Russel's paradox: I collected all things satisfying a sentence. Fear not, by the axioms of set theory this is one of the few allowable cases of this notation (In the previous notes we misused set-builder notation in the definition of unions. This is another allowed case).

Example 1.1 Let $A = \emptyset$. The power set of A is $\mathcal{P}(A) = \{\, \emptyset \,\}$. Do not confuse this set with the empty set. $\{\, \emptyset \,\}$ is the set that contains the empty set, and hence is not empty. ∎

Example 1.2 Let $A = \{\, 1,\, 2 \,\}$. The power set $\mathcal{P}(A)$ is:

$$\mathcal{P}(A) = \big\{\, \emptyset,\, \{\, 1 \,\},\, \{\, 2 \,\},\, \{1,\, 2 \,\} \,\big\} \tag{2}$$

∎

Note that for any set A, $\emptyset \subseteq A$ is true. So $\emptyset \in \mathcal{P}(A)$ is always true. Also, $A \subseteq A$ is true, so $A \in \mathcal{P}(A)$ is also true. We can visualize the power set of finite sets via *Hasse Diagrams* (Fig. 1).

The axioms of set theory also give us the *Cartesian product*. Kuratowski, one of the pioneers of point-set topology, tells us how we can define *ordered pairs*. Given a and b, we write:

$$(a, b) = \big\{\, \{\, a \,\},\, \{\, a,\, b \,\} \,\big\} \tag{3}$$

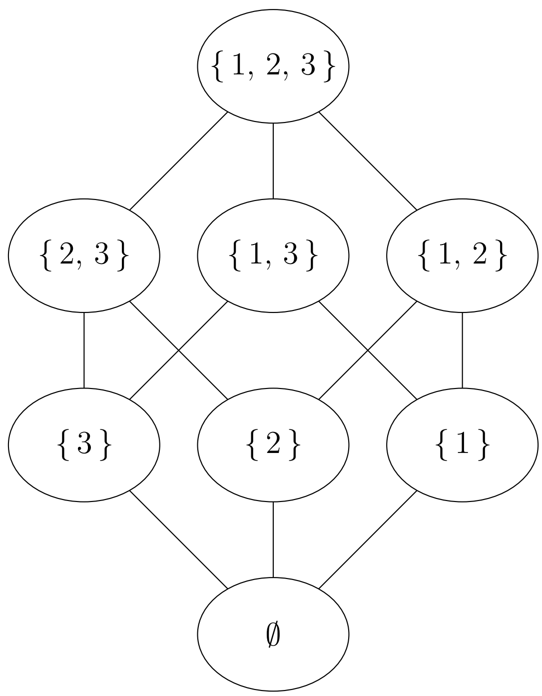

Figure 1: Power Set of { 1, 2, 3 }

This definition allows us to define ordered pairs using sets. It has the property that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. In particular, if a and b are distinct, then (a, b) and (b, a) are different objects. That is, ordered pairs have *order*.

Note that if A and B are sets, if $a \in A$, and if $b \in B$, then $(a,\, b)$ is an element of $\mathcal{P}\big(\mathcal{P}(A \cup B)\big)$. By collecting all elements of this set that are ordered pairs from A and B, we get the *Cartesian Product*.

Definition 1.2 (**Cartesian Product**) The Cartesian product of a set A with a set B is the set $A \times B$ defined by:

$$A \times B = \{\, (a,\, b) \mid a \in A \text{ and } b \in B \,\} \tag{4}$$

That is, the set of all ordered pairs $(a,\, b)$ with $a \in A$ and $b \in B$. ■

If we wanted to avoid abusing set-builder notation, we would write:

$$A \times B = \{\, (a,\, b) \in \mathcal{P}\big(\mathcal{P}(A \cup B)\big) \mid a \in A \text{ and } b \in B \,\} \tag{5}$$

but this is a bit messy.

Example 1.3 If $A = \{\, 1,\, 2 \,\}$ and $B = \{\, a,\, b \,\}$, the Cartesian product $A \times B$ is the set:

$$A \times B = \{\, (1,\, a),\, (1,\, b),\, (2,\, a),\, (2,\, b) \,\} \tag{6}$$

The Cartesian product $B \times A$ is slightly different:

$$B \times A = \{\, (a,\, 1),\, (a,\, 2),\, (b,\, 1),\, (b,\, 2) \,\} \tag{7}$$

In general, if A and B are different sets, then $A \times B$ and $B \times A$ are not equal. ■

Example 1.4 The Euclidean plane $\mathbb{R}^2$ is the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. ■

If we have two subsets A and B of the real line $\mathbb{R}$, we can visualize $A \times B$ as a subset of $\mathbb{R}^2$. This is done in Fig. 2. Points in A are shown in green, points in B in red, and the Cartesian product is in blue.

2 Functions

A function from a set A to a set B is a *rule* that assigns to each $a \in A$ a unique element $b \in B$. We could adopt the word function as a primitive, but it is possible to define functions precisely using our already developed vocabulary from set theory.

Definition 2.1 (**Function**) A function from a set A to a set B is a subset $f \subseteq A \times B$, denoted $f : A \to B$, such that for all $a \in A$ there is a unique $b \in B$ with $(a, b) \in f$. We write $b = f(a)$. ■

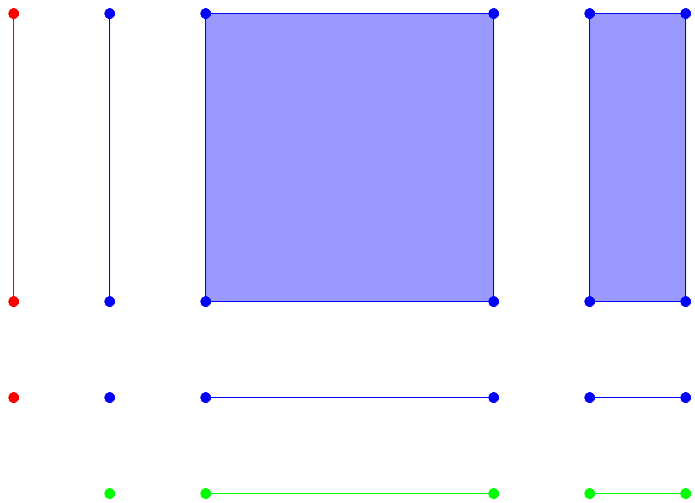

Figure 2: Cartesian Product of Sets in $\mathbb{R}$

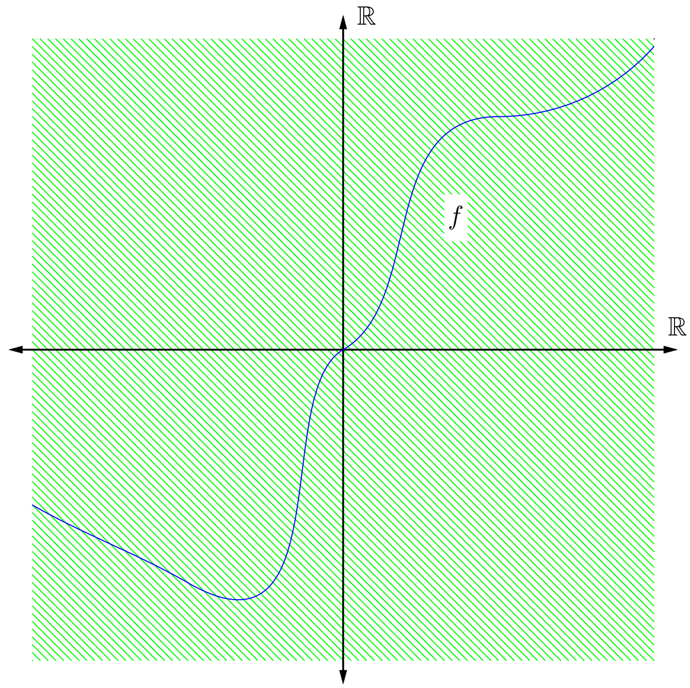

Figure 3: A Function $f : \mathbb{R} \to \mathbb{R}$

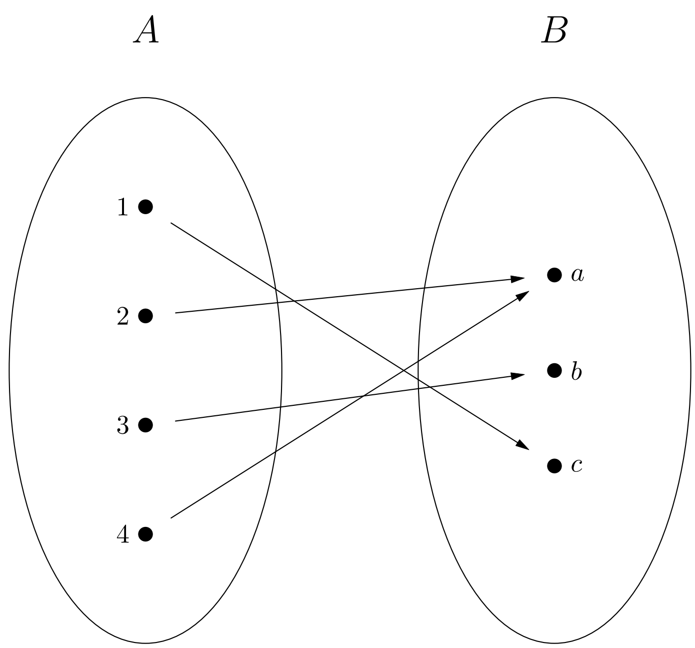

Figure 4: A Function from A to B

Fig. 3 shows this definition in action. The green denotes $\mathbb{R}^2$ and the blue curve that cuts through the plane is a subset $f \subseteq \mathbb{R} \times \mathbb{R}$. This is our usual notion of *function*, especially functions of a real variable.

Functions can be more abstract and do not need to be represented by curves in the plane. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. The diagram in Fig. 4 depicts a valid function $f : A \to B$. To each element in A there is a unique element in B it is assigned to. Contrast this with Figs. 5, 6, and 7.

There are three special types of functions.

Definition 2.2 (**Injective Function**) An injective function from a set A to a set B is a function $f : A \to B$ such that for all $x, y \in A$, $f(x) = f(y)$ if and only if $x = y$. ■

Example 2.1 The functions $f(x) = \sqrt{x}$ defined on $\mathbb{R}_{\geq 0}$, $\exp(x)$ defined on $\mathbb{R}$, and $\ln(x)$ defined on $\mathbb{R}^+$ are all injective. ■

Definition 2.3 (**Surjective Function**) A surjective function from a set A to a set B is a function $f : A \to B$ such that for all $b \in B$ there is an $a \in A$ with $f(a) = b$. ■

Example 2.2 The function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is surjective. Every real number $r \in \mathbb{R}$ corresponds to the tangent of some angle θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. ■

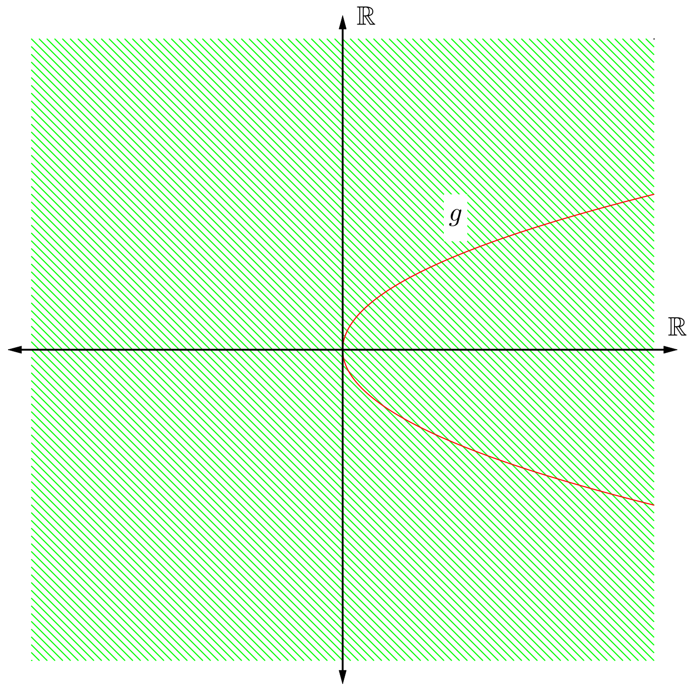

Figure 5: A Non-Function on ℝ

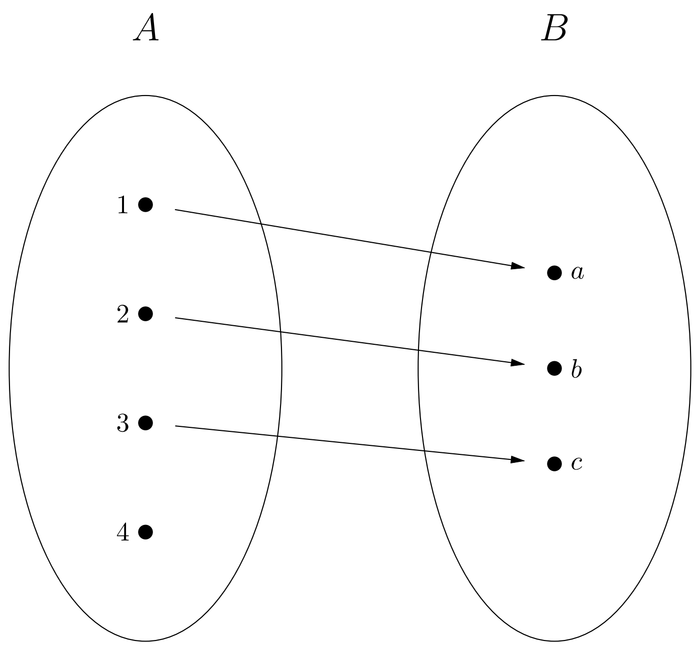

Figure 6: An Abstract Non-Function

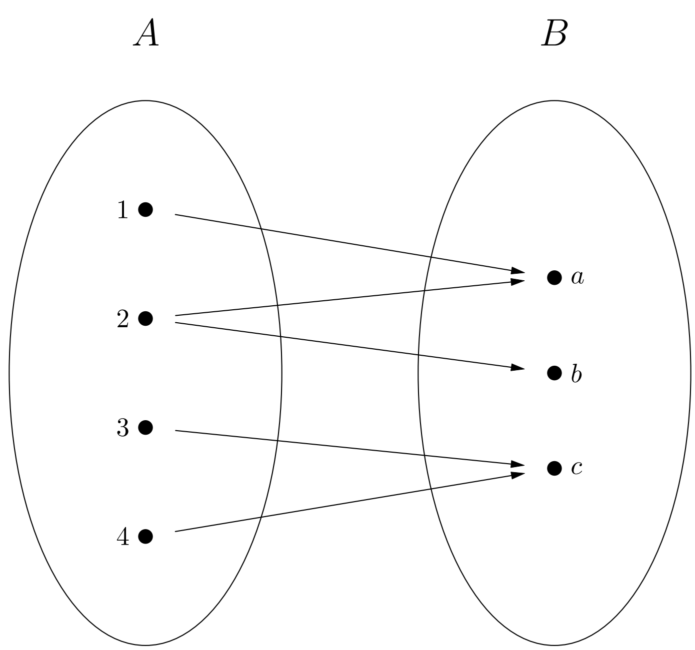

Figure 7: Another Non-Function

Example 2.3 The function $f(x) = x^3 - x$ is surjective, but *not* injective. It is surjective because it is continuous and as x tends to positive infinity, $f(x)$ tends to positive infinity as well. Similarly as x tends to negative infinity, so does $f(x)$. By the intermediate value theorem, f hits every value in between, meaning f is surjective. It is not injective since $f(0) = f(1) = 0$. ■

Definition 2.4 (**Bijective Function**) A bijective function from a set A to a set B is a function $f : A \to B$ such that f is injective and surjective. ■

Example 2.4 The functions $\exp : \mathbb{R} \to \mathbb{R}^+$, $\ln : \mathbb{R}^+ \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ are all bijective. ■

We've defined functions in previous examples using formulas. For example, we could define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. What is meant is the set $f \subseteq \mathbb{R} \times \mathbb{R}$ defined by:

$$f = \{\, (x,\, x^2) \in \mathbb{R} \times \mathbb{R} \mid x \in \mathbb{R} \,\} \tag{8}$$

In practice we do not define functions by sets like this, but rather by formulas. You must be careful that your formula is *well-defined*.

Example 2.5 Let $f : \mathbb{Q} \to \mathbb{Z}$ be defined by:

$$f\big(\frac{p}{q}\big) = p \tag{9}$$

Is this really a function? Let's look at $\frac{1}{2}$. We have:

$$f\big(\frac{1}{2}\big) = 1 \tag{10}$$

We also have:

$$f\big(\frac{1}{2}\big) = f\big(\frac{2}{4}\big) = 2 \tag{11}$$

so the formula f does not actually define a function, meaning our $f : \mathbb{Q} \to \mathbb{Z}$ notation is misleading. f fails to have the *uniqueness* property of functions. This is often called the *vertical line test* in calculus. ■

Definition 2.5 (**Composition**) The composition of functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$ defined by:

$$(g \circ f)(a) = g\big(f(a)\big) \tag{12}$$

That is, apply f to a first, and then apply g to $f(a)$. ■

Definition 2.6 (**Inverse Function**) An inverse of a function $f : A \to B$ is a function $g : B \to A$ such that $(g \circ f)(a) = a$ and $(f \circ g)(b) = b$ for all $a \in A$ and all $b \in B$. ■

Theorem 2.1. *A function $f : A \to B$ is bijective if and only if it has an inverse function $g : B \to A$.*

Proof. Suppose $f : A \to B$ is bijective. By definition, f is injective and surjective. That is, for each $b \in B$, there is an $a \in A$ such that $f(a) = b$ (surjectivity), and this $a \in A$ is unique (injectivity). Define $g : B \to A$ by setting $g(b)$ equal to the unique $a \in A$ with $f(a) = b$. By definition, $g\big(f(a)\big) = a$ and $f\big(g(b)\big) = b$, so g is an inverse function of f. In the other direction, suppose $f : A \to B$ has an inverse function $g : B \to A$. Suppose $x, y \in A$ are such that $f(x) = f(y)$. Then $g\big(f(x)\big) = g\big(f(y)\big)$. But $g\big(f(x)\big) = x$ and $g\big(f(y)\big) = y$ since g is an inverse of f, meaning $x = y$ and hence f is injective. If $b \in B$, let $a = g(b)$. Then $f(a) = f\big(g(b)\big) = b$, and hence f is surjective. Therefore, f is bijective. □

Theorem 2.2. *If $f : A \to B$ is a function, and if $g_0, g_1 : B \to A$ are inverses of f, then $g_0 = g_1$.*

Proof. Let $b \in B$. Since f has an inverse, it is bijective, and hence there is an $a \in A$ with $f(a) = b$. But then:

$$\begin{aligned} g_0(b) &= g_0\big(f(a)\big) && (13)\\ &= a && (14)\\ &= g_1\big(f(a)\big) && (15)\\ &= g_1(b) && (16) \end{aligned}$$

and hence $g_0(b) = g_1(b)$, so g_0 and g_1 are the same function. □

This tells us inverse functions are unique. We may then adopt the following notation.

Notation 2.1 If $f : A \to B$ is a function with an inverse $g : B \to A$ we denote this $g = f^{-1}$. ■

Definition 2.7 (**Image of a Set**) The image of a subset $\mathcal{U} \subseteq A$ of a function $f : A \to B$ is the set $f[\mathcal{U}] \subseteq B$ defined by:

$$f[\mathcal{U}] = \{\, b \in B \mid \text{there exists } a \in \mathcal{U} \text{ such that } f(a) = b \,\} \qquad (17)$$

That is, the set of all values $f(a)$ for all $a \in \mathcal{U}$. ■

Example 2.6 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. For every positive real number y there is a positive real number x such that $x^2 = y$, notably $x = \sqrt{y}$. Also, $0^2 = 0$. We also know that x^2 is always non-negative, so $x^2 < 0$ is never true for real numbers. We conclude that $f[\mathbb{R}] = \mathbb{R}_{\geq 0}$. ■

Definition 2.8 (**Pre-Image of a Set**) The pre-image of a subset $\mathcal{V} \subseteq B$ of a function $f : A \to B$ is the set $f^{-1}[\mathcal{V}] \subseteq A$ defined by:

$$f^{-1}[\mathcal{V}] = \{\, a \in A \mid f(a) \in \mathcal{V} \,\} \qquad (18)$$

That is, the set of all $a \in A$ whose image lies in $\mathcal{V}$. ■

Example 2.7 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(x)$. The pre-image of the set $(0, 1)$ is the set of all real numbers x with $0 < \sin(x) < 1$. This is the set of real numbers of the form $r = x + 2\pi n$ with $0 < x < \pi$ and $n \in \mathbb{Z}$. ■

Bijections allow us to define the size of sets. Two sets are said to have the same size, or same *cardinality*, if there is a bijection between them.

Notation 2.2 We use the notation $\mathbb{Z}_n$ to denote the set of integers 0, 1, ..., up to $n-1$, inclusive. ■

A finite set is a set that has a bijection with $\mathbb{Z}_n$ for some natural number $n \in \mathbb{N}$. An infinite set is a set that is not finite. The *smallest* infinity in set theory is the size of the natural numbers.

Theorem 2.3. *There is a bijection $f : \mathbb{N} \to \mathbb{Z}$.*

Proof. Define $f(n)$ by:

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n+1}{2} & n \text{ odd} \end{cases} \tag{19}$$

This is injective. If n and m are different numbers, and n is odd, and m is even, then $m/2$ and $-(n+1)/2$ yield different values. If both m and n are even, then $\frac{m}{2} = \frac{n}{2}$ if and only if $m = n$. Lastly if m and n are both odd, then $-\frac{m+1}{2} = -\frac{n+1}{2}$ if and only if $m = n$. This is also surjective. Given $N \in \mathbb{Z}$, $N \geq 0$, choose $n = 2N$. Then $f(n) = N$. If $N < 0$, choose $n = -1 - 2N$. Then $f(n) = N$. Note $-1 - 2N$ is positive since N is negative, meaning $f(n)$ is well-defined. This shows f is surjective, and since it is also injective, f is bijective. □

This means that $\mathbb{Z}$ is *countably infinite*. A countable set is either finite, or can be put into bijection with $\mathbb{N}$. More surprisingly, the rational numbers are countable. It is hard to find an explicit bijection between $\mathbb{N}$ and $\mathbb{Q}$. Instead, we invoke the *Cantor-Schröeder-Bernstein* theorems.

Theorem 2.4. *If A and B are sets, and if there exist injective functions $f : A \to B$ and $g : B \to A$, then there is a bijective function $h : A \to B$.*

Theorem 2.5. *If A and B are sets, and if there exist surjective functions $f : A \to B$ and $g : B \to A$, then there is a bijective function $h : A \to B$.*

We can describe a surjection $f : \mathbb{N} \to \mathbb{Q}^+$ via picture in Fig. 8. A surjection $f : \mathbb{N} \to \mathbb{Q}$ is given in Fig. 9.

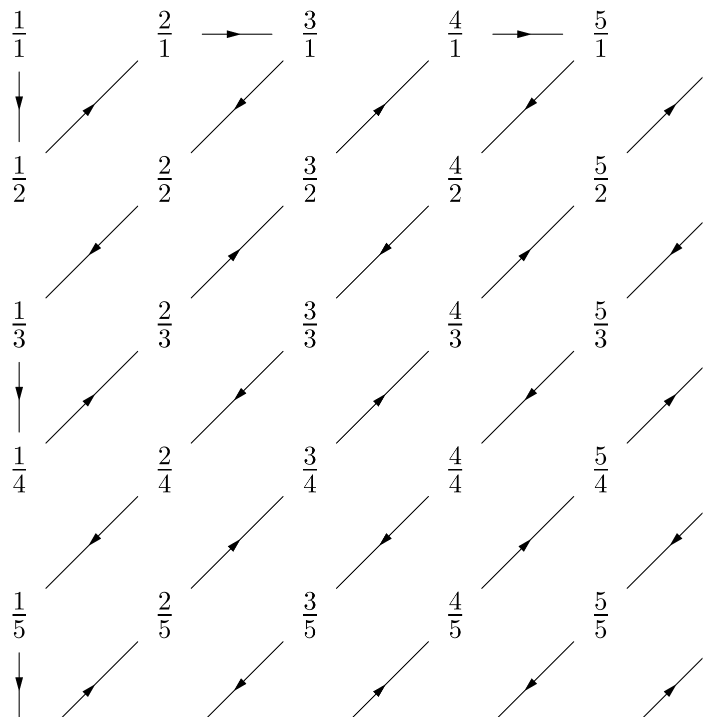

Figure 8: A Surjection from $\mathbb{N}$ to $\mathbb{Q}^+$

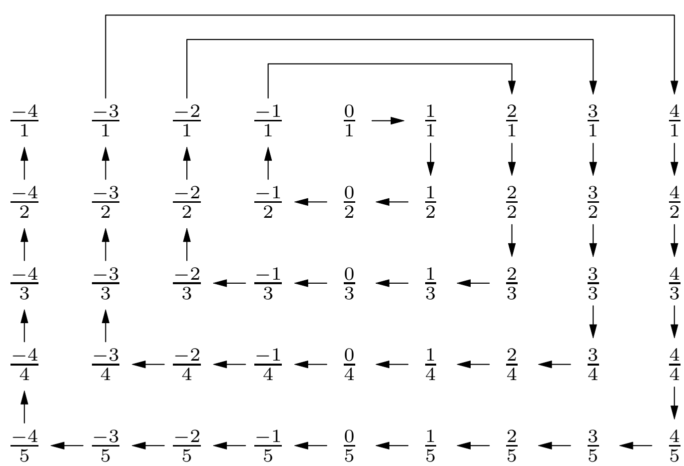

Figure 9: A Surjection from $\mathbb{N}$ to $\mathbb{Q}$