Point-Set Topology: Lecture 3

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1 More Cardinality

In the last lecture we showed $\text{Card}(\mathbb{N}) = \text{Card}(\mathbb{Z}) = \text{Card}(\mathbb{Q})$. A *countably infinite* set is a set that can be put into a bijection with $\mathbb{N}$. A *countable* set is a set that is either countably infinite or finite. An *uncountable* set is a set that is infinite but not countable. We now arrive at our first uncountable set, the real numbers $\mathbb{R}$. Suppose they are countable. Then there is a bijection $f : \mathbb{N} \to \mathbb{R}$. For simplicity, let us assume there is a bijection $f : \mathbb{N} \to (0, 1)$. Then we can write out this bijection with a list.

$$f(0) = 0.d_{0,0}d_{0,1}d_{0,2}d_{0,3}d_{0,4}d_{0,5}\dots \tag{1}$$
$$f(1) = 0.d_{1,0}d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}\dots \tag{2}$$
$$f(2) = 0.d_{2,0}d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}\dots \tag{3}$$
$$f(3) = 0.d_{3,0}d_{3,1}d_{3,2}d_{3,3}d_{3,4}d_{3,5}\dots \tag{4}$$
$$f(4) = 0.d_{4,0}d_{4,1}d_{4,2}d_{4,3}d_{4,4}d_{4,5}\dots \tag{5}$$
$$f(5) = 0.d_{5,0}d_{5,1}d_{5,2}d_{5,3}d_{5,4}d_{5,5}\dots \tag{6}$$

where $d_{m,n}$ is the decimal of the m^{th} number in the n^{th} decimal place. Since the bijection is between $\mathbb{N}$ and $(0, 1)$, the interger part of each $f(n)$ is zero. We now show that f is not a bijection by giving a new number that is not on the list. Define $r \in (0, 1)$ as follows:

$$r = 0.r_0r_1r_2r_3r_4r_5\dots \tag{7}$$

where

$$r_n = \begin{cases} d_{n,n} + 1 & d_{n,n} \neq 9 \\ 0 & d_{n,n} = 9 \end{cases} \tag{8}$$

This number is not equal to $f(n)$ for any n. It is not $f(0)$ since r_0 and $d_{0,0}$ are different. It is not $f(1)$ since r_1 and $d_{1,1}$ differ. Similarly, it is not $f(n)$ since r_n and $d_{n,n}$ are not the same decimal. So r is not on our list, meaning $f(n) \neq r$ for any $n \in \mathbb{N}$, contradicting the fact that f is a bijection.

There are small gaps here, meaning this is a *sketch of proof* and not a full proof. The argument does not take into account the fact that $0.1 = 0.0\overline{9}$, for example, but this can be corrected.

Theorem 1.1 (**Cantor's Theorem**). *If A is a set, then there is an injective function $f : A \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the power set of A, but there exists no surjection, and hence no bijection.*

Proof. Suppose there is a surjection $f : A \to \mathcal{P}(A)$. Define $B \subseteq A$ by:

$$B = \{\, x \in A \mid x \notin f(x) \,\} \tag{9}$$

Since $f(x) \in \mathcal{P}(A)$ for all $x \in A$, it is valid to ask if $x \in f(x)$. Since $B \subseteq A$ we have $B \in \mathcal{P}(A)$ by the definition of power set. But since $f : A \to \mathcal{P}(A)$ is a surjection there must be some $y \in A$ such that $f(y) = B$. But then either $y \in B$ or $y \notin B$. Suppose $y \in B$. If $y \in B$, then $y \in f(y)$ since $f(y) = B$. But if $y \in B$, then by the definition of B that means $y \notin f(y)$, a contradiction. So $y \notin B$. But if $y \notin B$, then $y \notin f(y)$ since $f(y) = B$. But by the definition of B, if $y \notin f(y)$, then $y \in B$, a contradiction. So $f(y) \neq B$, and hence f is not a surjection.

There is an injective function $f : A \to \mathcal{P}(A)$. Define:

$$f(x) = \{\, x \,\} \tag{10}$$

Then $f(x) = f(y)$ if and only if $\{\, x \,\} = \{\, y \,\}$, which is true if and only if $x = y$, hence f is injective. □

There is a bijection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$. Again, a sketch of proof is given. We'll construct a surjection $f : \mathcal{P}(\mathbb{N}) \to [0,1]$, the closed unit interval. Given a set $A \subseteq \mathbb{N}$, construct the number $r \in [0,1]$ using binary as follows:

$$f(A) = r = 0.r_0 r_1 r_2 \dots \tag{11}$$

where $r_n = 1$ if $n \in A$ and $r_n = 0$ if $n \notin A$. For example, if $A = \emptyset$, then $f(\emptyset) = 0.000\cdots = 0$. If $A = \mathbb{N}$, then $f(\mathbb{N}) = 0.111\cdots = 1$. If $A = \{\, 0,\, 2,\, 4,\, \dots \}$, then $f(A) = 0.101010\dots$. If $A = \{\, 1,\, 2,\, 3 \,\}$, then $f(A) = 0.01110000\dots$. Since every number $r \in [0,1]$ can be written in binary form in such a way, f is a surjection. We can reverse this process as well, but again the issue of 1 vs. $0.\overline{9}$ arises and needs correcting. It is possible to do this, but not currently worth our time investigating.

You may now ask *this is a bijection from the natural numbers to the closed unit interval. What about* $\mathbb{R}$? We can construct a bijection $g : [0,1] \to (0,1)$, the closed unit interval to the open unit interval, via:

$$g(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{x}{4} & x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases} \tag{12}$$

The graph is shown in Fig. 1. We will eventually prove that there is no *continuous* bijection $f : [0,1] \to (0,1)$. For those interested, try to find a bijection $f : [0,1] \to (0,1)$ that has only *finitely many* discontinuities.

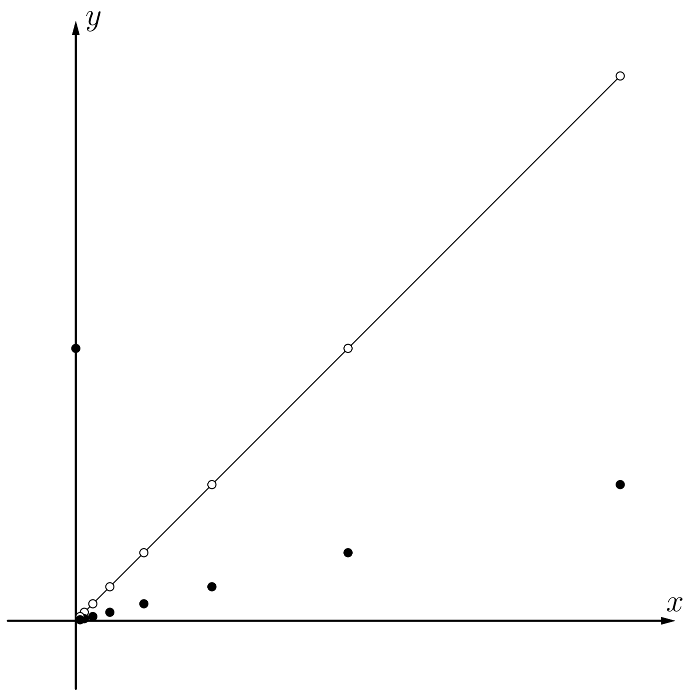

Figure 1: Bijection from $[0, 1]$ to $(0, 1)$

Using this bijection g, we need a bijection from $(0, 1)$ to $\mathbb{R}$. This is given by:

$$h(x) = \frac{2x - 1}{x(1 - x)} \tag{13}$$

By composing $h \circ g \circ f$ we get a bijection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$. This means that cardinality is *transitive*.

Theorem 1.2. *If Card$(A) =$ Card(B) and Card$(B) =$ Card(C), then Card$(A) =$ Card(C).*

Proof. Since A and B are of the same cardinality, there is a bijection $f : A \to B$. Similarly, there is a bijection $g : B \to C$. By composing we get a bijection $g \circ f : A \to C$, meaning $\text{Card}(A) = \text{Card}(C)$. □

2 Relations

Relations are ways of saying certain elements of a set are related to each other. There are many relations you use daily in mathematics. Equality $(=)$, less than $(<)$, greater than $(>)$, less than or equal $(\leq)$, and greater than or equal $(\geq)$.

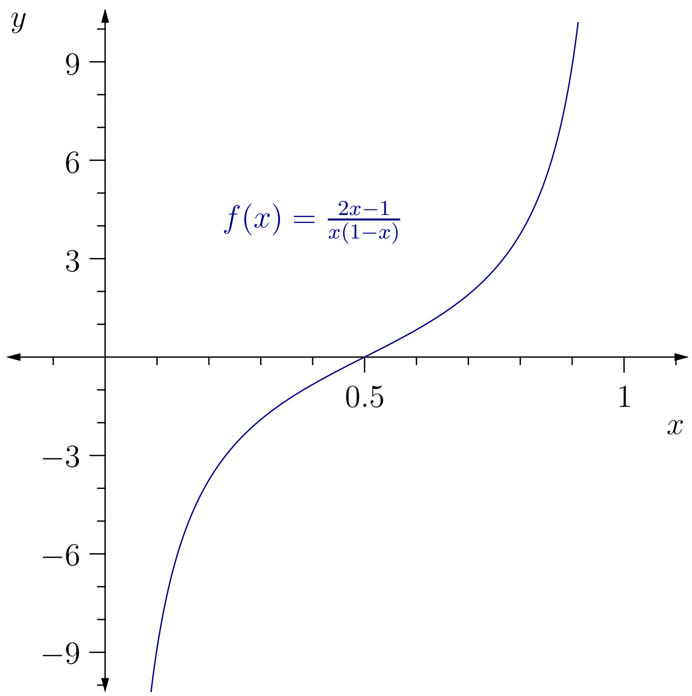

Figure 2: Bijection from $(0, 1)$ to $\mathbb{R}$

We've also seen relations on sets such as *inclusion* ($\subseteq$) and *proper inclusion* ($\subsetneq$). Cardinality can also be thought of as a type of relation on sets. The most general definition of a relation is as follows.

Definition 2.1 (**Relation**) A relation on a set A is a subset $R \subseteq A \times A$. ■

If $(a, b) \in R$ we write this as aRb.

Example 2.1 Suppose we know what *less than* means for real numbers. We can define $<$ to be the set:

$$< = \{ (a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b \} \tag{14}$$

Rather than writing $(a, b) \in <$, we write $a < b$. It's weird to think of the symbol $<$ as a set, and in practice we don't. We think of it as a way of relating elements in $\mathbb{R}$. Similarly, for a set A and a relation R, you should think of R as a way of relating elements. ■

Example 2.2 The natural numbers can be given a precise construction. We write $0 = \emptyset$, $1 = \{ 0 \}$, $2 = \{ 0, 1 \}$, $3 = \{ 0, 1, 2 \}$, and so on. We can now define $<$ on $\mathbb{N}$ as follows:

$$< = \{ (m, n) \in \mathbb{N} \times \mathbb{N} \mid m \in n \} \tag{15}$$

This is bizarre, but makes precise what inequality means. Since $3 = \{ 0, 1, 2 \}$, we see that $1 \in 3$, meaning we can write $1 < 3$. This is in agreement with the way we intuitively think of the *less than* relation. ■

Example 2.3 If X is a set, and $R \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ is defined by:

$$R = \{ (A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \subseteq B \} \tag{16}$$

then R is the *inclusion* relation on the set of all subsets of X. ■

Since the definition of relation is so general (*any* subset of $A \times A$), it is often the case the we restrict our attention to special types of relations.

Definition 2.2 (**Reflexive Relation**) A reflexive relation on a set A is a relation R such that for all $a \in A$, aRa. That is, for all $a \in A$, a is related to itself by R. ■

Example 2.4 Equality ($=$) is reflexive, as is inclusion ($\subseteq$). ■

Example 2.5 Proper inclusion ($\subsetneq$) is not reflexive, neither is less than ($<$) nor greater than ($>$). ■

Given a set A, the *diagonal* of $A \times A$ is the set of all ordered pairs (a, a) for all $a \in A$. If we look at $\mathbb{R} \times \mathbb{R}$, the diagonal is the line $y = x$ in the plane, hence the name. A reflexive relation is a relation R that contains the diagonal.

Definition 2.3 (**Symmetric Relation**) A symmetric relation on a set A is a relation R such that for all $a, b \in A$, aRb if and only if bRa. ■

Example 2.6 Equality is symmetric. $a = b$ implies $b = a$. ■

Example 2.7 Containment ($\in$) is not symmetric. It is a theorem of set theory that $x \in y$ implies $y \notin x$. The importance of this claim is that it helps us avoid Russell's paradox, one of the reasons for developing set theory in the first place. ■

Example 2.8 Inclusion is a relation that is reflexive but not symmetric. It is possible for $A \subseteq B$ but $B \not\subseteq A$. For example, $A = \mathbb{Q}$ and $B = \mathbb{R}$. ■

Definition 2.4 (**Transitive Relation**) A transitive relation on a set A is a relation R such that for all $a, b, c \in A$, aRb and bRc implies aRc. ■

Example 2.9 Equality is transitive. This is one of the assumptions dating back to Euclid. If $a = b$ and $b = c$, then $a = c$. ■

Example 2.10 Inequality is also transitive. If $a < b$ and $b < c$, then $a < c$. ■

Example 2.11 Inclusion is transitive. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. ■

Example 2.12 Containment does not need to be transitive. Let $a = \emptyset$, $b = \{\,\emptyset\,\}$, and $c = \{\,\{\,\emptyset\,\}\,\}$. Then $a \in b$, $b \in c$, but $a \notin c$. ■

Definition 2.5 (**Equivalence Relation**) An equivalence relation on a set A is a relation R that is reflexive, symmetric, and transitive. ■

Equivalence relations allow us to define equivalence classes.

Definition 2.6 (**Equivalence Class**) The equivalence class of an element $a \in A$ with respect to an equivalence relation R is the set $[a]$ defined by:

$$[a] = \{\, b \in A \mid aRb \,\} \tag{17}$$

That is, the set of all elements in A related to a by R. ■

Theorem 2.1. *If A is a set, if R is an equivalence relation, and if $a, b \in A$, then $[a] = [b]$ if and only if aRb and bRa.*

Proof. Since R is reflexive, $a \in [a]$ and $b \in [b]$. If aRb, then $b \in [a]$, by definition. But R is symmetric, so bRa and hence $a \in [b]$. That is, the sets $[a]$ and $[b]$ both contain a and b. If $c \in [a]$ then aRc. But bRa, and since R is transitive, bRc. Therefore $c \in [b]$. Similarly, $c \in [b]$ implies $c \in [a]$. We have shown that $[a]$ and $[b]$ consist of precisely the same elements, so $[a] = [b]$. In the other direction, if $[a] = [b]$, then by definition $a \in [b]$ and $b \in [a]$, and hence aRb and bRa. □

Definition 2.7 (**Quotient Set**) The quotient set of a set A with respect to an equivalence relation R is the set A/R defined by:

$$A/R = \{\, B \in \mathcal{P}(A) \mid B = [a] \text{ for some } a \in A \,\} \tag{18}$$

That is, A/R is the set of all equivalence classes of A with respect to R. ■

The notation A/R is just notation. We are not *dividing* sets. Intuitively, we are forming a new set by taking all of the elements $b \in A$ such that $b \in [a]$ and *gluing* them to a, creating a single element. This will be very important in topology when we talk about *quotient spaces*.

Example 2.13 We can think of a fraction $\frac{a}{b}$ as an ordered pair $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{ 0 \})$. We do not want $\frac{1}{2}$ and $\frac{2}{4}$ to be different elements, so we need to *glue* some elements of this product together. That is, $\mathbb{Z} \times (\mathbb{Z} \setminus \{ 0 \})$ is a set of points in the plane $\mathbb{R}^2$ and the points (1, 2) and (2, 4) are different. We ask *how can we say* $\frac{a}{b} = \frac{c}{d}$ *using only integers*? We are trying to define what a rational number is, so it would be circular to use the notation $\frac{a}{b}$ in our argument. We obtain the answer via cross-multiplying. We know that $\frac{a}{b} = \frac{c}{d}$ is true (essentially by definition) when $ad = bc$. This allows us to define an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{ 0 \})$. Let $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{ 0 \})$ and define

$$R = \{ \big((a,\, b),\, (c,\, d)\big) \in A \times A \mid ad = bc \} \tag{19}$$

The quotient set A/R is the set of *rational numbers*. The equivalence classes $[(1,\, 2)]$ and $[(2,\, 4)]$ are the same since $1 \cdot 4 = 2 \cdot 2$. That is, we have glued together (1, 2) and (2, 4) to form a single object, the fraction $\frac{1}{2}$. We write $[(a,\, b)] = \frac{a}{b}$ for convenience. ■