Point-Set Topology: Lecture 4

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1 Generalized Operations on Sets

In topology, set theory, analysis, etc., we often need to consider more than just two sets at a time. We may have a set X and a set $\mathcal{O}$ consisting of subsets $A \subseteq X$. The union over such a collection exists by the axiom of the union. The definition of this set is:

Definition 1.1 (**Union of a Collection**) The union over a set $\mathcal{O}$ is the set $\bigcup \mathcal{O}$ defined by:

$$\bigcup \mathcal{O} = \{\, x \mid \text{there exists } A \in \mathcal{O} \text{ such that } x \in A \,\} \tag{1}$$

That is, the set of all elements contained in some set $A \in \mathcal{O}$. ■

Note we've once again abused set-builder notation. The existence of unions is one of the axioms of set theory, and this allows us to write such equations.

The intersection does not need an axiom, and can be defined via the union set and the axiom schema of specification.[1]

Definition 1.2 (**Intersection of a Collection**) The intersection over a set $\mathcal{O}$ is the set $\bigcap \mathcal{O}$ defined by:

$$\bigcap \mathcal{O} = \{\, x \in \bigcup \mathcal{O} \mid x \in A \text{ for all } A \in \mathcal{O} \,\} \tag{2}$$

That is, the set of all elements common to all $A \in \mathcal{O}$. ■

If the set $\mathcal{O}$ is indexed by the natural numbers, say:

$$\mathcal{O} = \{\, A_0,\, A_1,\, A_2,\, \dots \} \tag{3}$$

We write the intersection and union as follows:

$$\bigcup \mathcal{O} = \bigcup_{n=0}^{\infty} A_n \tag{4}$$

$$\bigcap \mathcal{O} = \bigcap_{n=0}^{\infty} A_n \tag{5}$$

[1] This is the *set-builder* axiom that says writing sets in the form $\{\, x \in X \mid P(x) \,\}$ is legal.

We may also write:

$$\bigcup \mathcal{O} = \bigcup_{A \in \mathcal{O}} A \tag{6}$$

$$\bigcap \mathcal{O} = \bigcap_{A \in \mathcal{O}} A \tag{7}$$

If $\mathcal{O}$ is indexed by some *indexing set* I (for example, the real numbers), where $A_\alpha \in \mathcal{O}$ for all $\alpha \in I$, we write:

$$\bigcup \mathcal{O} = \bigcup_{\alpha \in I} A_\alpha \tag{8}$$

$$\bigcap \mathcal{O} = \bigcap_{\alpha \in I} A_\alpha \tag{9}$$

The *generalized distributive laws* then say:

$$A \cap \bigcup_{B \in \mathcal{O}} B = \bigcup_{B \in \mathcal{O}} (A \cap B) \tag{10}$$

$$A \cup \bigcap_{B \in \mathcal{O}} B = \bigcap_{B \in \mathcal{O}} (A \cup B) \tag{11}$$

The *generalized De Morgan's Laws* read:

$$X \setminus \bigcup_{A \in \mathcal{O}} A = \bigcap_{A \in \mathcal{O}} (X \setminus A) \tag{12}$$

$$X \setminus \bigcap_{A \in \mathcal{O}} A = \bigcup_{A \in \mathcal{O}} (X \setminus A) \tag{13}$$

2 Product Sets and the Axiom of Choice

My goal, now, is to convince you there is another way of thinking of Cartesian products. A *better* way. The axioms of set theory tell us there exists a *set of all functions* from a set A to a set B. That is, given sets A and B, there is a set $\mathcal{F}(A,\, B)$ such that for all f, $f \in \mathcal{F}(A,\, B)$ if and only if $f : A \to B$ is a function from A to B. This can be proven using the axiom of the power set and of specification, but we won't bother. Let us redefine the Cartesian product of a set A with a set B to be the set of all functions $f : \mathbb{Z}_2 \to A \cup B$ with the property that $f(0) \in A$ and $f(1) \in B$. Take $\mathbb{R}^2$ for example. An element of $\mathbb{R}^2$ is then a function $\mathbf{x} : \mathbb{Z}_2 \to \mathbb{R} \cup \mathbb{R}$, which we may simply write $\mathbf{x} : \mathbb{Z}_2 \to \mathbb{R}$, such that $\mathbf{x}(0) \in \mathbb{R}$ and $\mathbf{x}(1) \in \mathbb{R}$. Let's rewrite $\mathbf{x}(0) = x_0$ and $\mathbf{x}(1) = x_1$. Then the *function* $\mathbf{x}$ is really just the *vector* $\mathbf{x} = (x_0,\, x_1)$. Why might we do this? It allows us to more easily define the product of three sets. $A \times (B \times C)$ and $(A \times B) \times C$ are different sets. $A \times (B \times C)$ consists of elements of the form $\big(a,\, (b,\, c)\big)$, whereas elements of $(A \times B) \times C$ are of the form $\big((a,\, b),\, c\big)$. Very nit-picky, but alas, this is how the Cartesian product was defined. Instead, we can define the product over three sets A, B, and C to be the set of all functions $f : \mathbb{Z}_3 \to A \cup B \cup C$ with the property that $f(0) \in A$, $f(1) \in B$, and $f(2) \in C$. We can say that this function is an *ordered triple* and write $f = \big(f(0),\, f(1),\, f(1)\big)$, which is more in line with our usual notion of higher order products. The main benefit is that we may define *arbitrary* products. If we have a set $\mathcal{O}$, we may define the *product set*.

Definition 2.1 (**Product Set**) The product set over a set $\mathcal{O}$ is the set $\prod \mathcal{O}$ defined by:

$$\prod \mathcal{O} = \{\, f : \mathcal{O} \to \bigcup \mathcal{O} \mid f(A) \in A \text{ for all } A \in \mathcal{O} \,\} \tag{14}$$

■

Intuitively, this is the set of all *generalized ordered tuples* $(a_0,\, a_1,\, a_2,\, \dots)$. The problem with this notation is it seems to imply the product is over a countable set, but it need not be. We can have *uncountably long* ordered tuples. The ordered tuple notation is just for convenience. The definition given above is rigorous. If $\mathcal{O}$ is indexed by the natural numbers, we can write:

$$\prod \mathcal{O} = \prod_{n=0}^{\infty} A_n = \{\, f : \mathcal{O} \to \bigcup_{n=0}^{\infty} A_n \mid f(A_n) \in A_n \text{ for all } n \in \mathbb{N} \,\} \tag{15}$$

We may also write:

$$\prod \mathcal{O} = \prod_{A \in \mathcal{O}} A = \{\, f : \mathcal{O} \to \bigcup_{A \in \mathcal{O}} A \mid f(A) \in A \text{ for all } A \in \mathcal{O} \,\} \tag{16}$$

Lastly, if $\mathcal{O}$ is indexed by some indexing set I, we can write:

$$\prod \mathcal{O} = \prod_{\alpha \in I} A_\alpha = \{\, f : \mathcal{O} \to \bigcup_{\alpha \in I} A_\alpha \mid f(A_\alpha) \in A_\alpha \text{ for all } \alpha \in I \,\} \tag{17}$$

Now for some controversy. If $\mathcal{O}$ is a non-empty set such that for all $A \in \mathcal{O}$ it is true that A is not empty, is $\prod \mathcal{O}$ empty? Let's suppose $\mathcal{O} = \{\, A,\, B \,\}$. Suppose $a \in A$ and $b \in B$. The product $\prod \mathcal{O}$ then consists of the element $f : \mathcal{O} \to A \cup B$ such that $f(A) = a$ and $f(B) = b$, so in particular, $\prod \mathcal{O}$ is not empty. Hurray! Indeed, if $\mathcal{O}$ is any *finite* set, $\prod \mathcal{O}$ is non-empty, by a similar argument. What if $\mathcal{O}$ is countable? We can write each set as A_n for some $n \in \mathbb{N}$. Since each A_n is non-empty, there is an element $a_n \in A_n$. We can then define the function $f : \mathcal{O} \to \bigcup_{n=0}^{\infty} A_n$ by $f(A_n) = a_n$. But hold on! What axiom allows us to define such a function? In the finite case, the axioms of set theory can prove the function given previously exists, but what about this infinite case? As it turns out, the existence of this function can *not* be proven using the axioms of set theory that we've so far discussed (union, pairing, power set, specification, infinity, extensionality, and the other two we have not mentioned). The claim is *independent* of these axioms. It is impossible to prove it is true and it is impossible to prove it is false. Contrast this with Euclidean geometry. The axioms are:

1. Given two points, there is a line segment between them.

2. Given a line segment, there is a line extending it on either side.

3. Given a point and a length, there is a circle with the point as its center and the length as its radius.

4. All right angles are equal in size.

5. If two straight lines fall on a third line, and if the sum of the angles the two straight lines make with the third sum to less than π, then the two straight lines intersect.

As mentioned a few classes ago, the fifth axiom is equivalent to Playfair's axiom.

5a.) Given a straight lines and point not on that line, there is a unique parallel line passing through the point.

For 2100 years many mathematicians (several quite rudely) attempted to prove the fifth axiom is redundant. That is, to prove the fifth axiom can be proven using the first four. In the 1800's C.E. it was shown that the fifth axiom is *independent* of the first four, meaning it cannot be proven or disproven. The *proofs* other mathematicans had that the fifth axiom can be proven simply introduced new assumptions (Khayyam introduced the quadrilateral axiom, Playfair introduced his axiom, and so on). There are two *models* of geometry that obey the first four but not the fifth. In hyperbolic geometry, given a point and a line, there are infinitely many parallel lines passing through the point. In spherical geometry there are no parallel lines.[2]

Just like Euclid's fifth axiom can not be proven from the other four, the axiom of choice can not be proven from the other axioms of set theory. Now you may say, *oh come on, obviously that set is not empty. In fact, that set is probably huge!* The *construction* given above certainly seems convincing, and this is why most accept the *axiom of countable choice.*

Axiom 2.1 (**Axiom of Countable Choice**) If $\mathcal{O}$ is a countable non-empty set such that for all $A \in \mathcal{O}$ it is true that A is non-empty, then $\prod \mathcal{O}$ is non-empty. ■

Richard Dedekind gave another definition of infinite. He says that an infinite set is a set A such that there is a proper subset $B \subsetneq A$ with $\mathrm{Card}(A) = \mathrm{Card}(B)$. This is now called *Dedekind infinite.* A set is *Dedekind finite* if it is not Dedekind infinite. The question is: Is A finite if and only if A is Dedekind finite? When you think about this question, it may eventually become *obvious* that these are the same definition. You may say *if I through a single element away, I end*

[2] Not too important, but spherical geometry does not obey the third axiom as it is worded here. It obeys the axiom that *given two points, there is a circle with one point as the center, and the other lying on the circle.*

up with a smaller set and hence the set must be finite! Without the axiom of countable choice, it is not possible to prove this statement.

Let's use the axiom of countable choice to prove that every subset of a countable set is countable. This comes in two steps.

Theorem 2.1. *If X is an infinite set, then there is a countably infinite subset $A \subseteq X$.*

Proof. Since X is infinite, for all $n \in \mathbb{N}$ the set $\mathcal{A}_n$ defined by:

$$\mathcal{A}_n = \{\, Y \subseteq X \mid \mathrm{Card}(Y) = 2^n \,\} \tag{18}$$

is non-empty (otherwise X would be finite). By the axiom of countable choice, the product $\prod_{n\in\mathbb{N}} \mathcal{A}_n$ is non-empty. Let $A : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ be an element of $\prod_{n\in\mathbb{N}} \mathcal{A}_n$. That is, for all $n \in \mathbb{N}$ we have that $A_n \in \mathcal{A}_n$. This means that for each $n \in \mathbb{N}$ the set A_n is a subset of X with 2^n elements. Define B_n via:

$$B_n = \bigcup_{k=0}^{n} A_k \tag{19}$$

By a counting argument we have that $\mathrm{Card}(B_n) < \mathrm{Card}(A_{n+1})$. That is:

$$\begin{aligned}
\mathrm{Card}(B_n) &\le \sum_{k=0}^{n} \mathrm{Card}(A_k) && (20)\\
&= \sum_{k=0}^{n} 2^k && (21)\\
&= 2^{n+1} - 1 && (22)\\
&< 2^{n+1} && (23)\\
&= \mathrm{Card}(A_{n+1}) && (24)
\end{aligned}$$

And hence the set C_n defined by:

$$C_n = A_{n+1} \setminus B_n \tag{25}$$

is non-empty. Moreover, from this construction we see that C_m and C_n are disjoint whenever $n \neq m$. By the axiom of countable choice the product set $\prod_{n\in\mathbb{N}} C_n$ is non-empty. Let $a : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} C_n$ be an element of the product. That is, for each $n \in \mathbb{N}$ we have that $a_n \in C_n$. Then $a : \mathbb{N} \to X$ is injective. That is, if $a_n = a_m$, then $a_n \in C_m$ and $a_m \in C_n$, meaning C_n and C_m are not disjoint. But this can only happen if $n = m$, and therefore $a : \mathbb{N} \to X$ is injective. The set $Y \subseteq X$ defined by:

$$Y = \{\, a_n \in X \mid n \in \mathbb{N} \,\} \tag{26}$$

is hence a countably infinite subset with $a : \mathbb{N} \to Y$ being a bijection. $\square$

Theorem 2.2. *If X is a countable set, and if $A \subseteq X$, then A is countable.*

Proof. If X is finite, then we are done since A must be finite as well. Suppose X is countably infinite. If A is finite, again we are done. So suppose A is infinite. Since X is countably infinite, there is a bijection $x : \mathbb{N} \to X$. Since A is infinite, there is a countably infinite subset $B \subseteq A$. But since B is countably infinite, there is a bijection $b : \mathbb{N} \to B$. But then the function $x^{-1} : X \to \mathbb{N}$ restricted to A is injective, and the function $b : \mathbb{N} \to A$ is also injective. By the Cantor-Schröeder-Bernstein theorem, there is a bijection $h : \mathbb{N} \to A$, and hence A is countable. $\square$

It would be quite strange if there was a set with size between finite and countable, which is one of the reasons the axiom of countable choice is accepted without much controversy. It also allows use to prove *Dedekind's theorem*, which was alluded to earlier.

Theorem 2.3 (**Dedekind's Theorem**)**.** *If X is a set, then it is infinite if and only if there is a proper subset $A \subsetneq X$ that has the same cardinality as X.*

Proof. If X has a proper subset $A \subsetneq X$ with the same cardinality as X, then X can not be finite. That is, if A has just one element missing from X, and if $\mathrm{Card}(X) = n$, then $\mathrm{Card}(A) \leq n - 1 < n$, and hence there can be no bijection between A and X.

The other direction requires the axiom of countable choice. We will use it indirectly by invoking a previous theorem. Suppose X is an infinite set. Then there is a countably infinite subset $B \subseteq X$. Let $b : \mathbb{N} \to B$ be a bijection. Define $f : X \to X$ via:

$$f(x) = \begin{cases} b_{n+1} & x = b_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases} \tag{27}$$

That is, f shifts the elements b_n along by 1 and leaves all other points fixed. Let $A = f[X]$. Then A is not equal to X. That is, $A \subsetneq X$. To see this, note that $b_0 \notin A$ since $f(x) \neq b_0$ for any $x \in X$ by the definition of f. However, the cardinality of A is the same as X. We realize this by finding a bijection between A and X. The function $f : X \to A$ is the bijection. To prove it is a bijection, we just need to show that it has an inverse function. This is given by:

$$f^{-1}(x) = \begin{cases} b_{n-1} & x = b_n \text{ for some } n \in \mathbb{N}^+ \\ x & \text{otherwise} \end{cases} \tag{28}$$

Hence $\mathrm{Card}(A) = \mathrm{Card}(X)$ and $A \subsetneq X$. $\square$

The general axiom of choice is not so embraced.

Axiom 2.2 (**Axiom of Choice**) If $\mathcal{O}$ is a non-empty set such that for all $A \in \mathcal{O}$ it is true that A is non-empty, then $\prod \mathcal{O}$ is non-empty. $\blacksquare$

Again, you might be screaming *of course that set is non-empty, it's probably uncountable!* This is impossible to prove, however. So, we may accept it as true and we may accept it as false. For this course, we will accept it as true. Here are a few equivalent statements that make it *obvious* that the statement is true.

Theorem 2.4. *If $f : A \to B$ is surjective, then there is a right inverse, a function $g : B \to A$ such that $(f \circ g)(b) = b$ for all $b \in B$.*

Proof. Since f is surjective, for all $b \in B$, there is an $a \in A$ such that $f(a) = b$. *Choose* $g(b) = a$. $\square$

We are not allowed to do this *choosing* without the axiom of choice. Indeed, if you think this proof should be valid, you are accepting the axiom of choice. This statement is *equivalent* to the axiom of choice.

Theorem 2.5. *If A and B are sets, either there is a surjection $f : A \to B$ or a surjection $g : B \to A$.*

Again, this is equivalent to choice. There are some algebraic equivalents.

- Every vector space has a basis.
- Every set has a group structure.
- Every ring has a maximal ideal.

In this course we will prove two topological facts that are also equivalent to the axiom of choice.

- The product of compact sets is compact.
- The product of connected sets is connected.

Lastly, an important theorem about quotient sets.

Theorem 2.6. *If A is a set, and R is an equivalence relation on A, then there is an injective function $f : A/R \to A$ such that $f(\mathcal{U}) \in \mathcal{U}$ for all $\mathcal{U} \in A/R$. Given $\mathcal{U} \in A/R$ and $f(\mathcal{U}) = x$, we write $\mathcal{U} = [x]$ and call this a representative for $\mathcal{U}$.*

Proof. For all $a \in A$, $[a]$ is non-empty set since $a \in [a]$. By the axiom of choice, there is a function $f : A/R \to \bigcup A/R$ such that for all $\mathcal{U} \in A/R$, $f(\mathcal{U}) \in \mathcal{U}$. But $\bigcup A/R = A$ (Remember, A/R is a set of subsets of A, the equivalence classes of A), so f is such a function from A/R to A. It is indeed injective. If $\mathcal{U}$ and $\mathcal{V}$ are equivalence classes, then $f(\mathcal{U}) = f(\mathcal{V})$ implies $\mathcal{U} = \mathcal{V}$ since $f(\mathcal{U}) \in \mathcal{V}$ and $f(\mathcal{V}) \in \mathcal{U}$ must be true. Hence, f is injective. $\square$

This is another intuitively *obvious* claim. *Of course I can pick a representative of an equivalence class!*

As you study algebra and topology more, you may become convinced that these statements *must* be true, and then you will become a defender of the axiom of choice. So why would anyone say it isn't true?

Theorem 2.7 (**Banach-Tarski Paradox**). *It is possible to take a sphere, cut it into five disjoint subsets, and using only rotation and translation, put the pieces back together into two new spheres, both being the same size as the original.*

The proof relies on the axiom of choice. Reading this statement over a few times you may think *that's clearly false.*

The big theorem that is *clearly false* we will need to use a few times. It is also equivalent to the axiom of choice. First, a few definitions.

Definition 2.2 (**Partial Order**) A partial order on a set A is a relation R such that R is reflexive, transitive, and if aRb and bRa, then $a = b$. ■

The two canonical examples are less than or equal to ($\leq$) and inclusion. $a \leq a$ is true. $a \leq b$ and $b \leq c$ implies $a \leq c$ is also true. Lastly, if $a \leq b$ and $b \leq a$, then $a = b$. Inclusion has the same properties. $A \subseteq A$, $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$, and $A \subseteq B$ and $B \subseteq A$ implies $A = B$. Inclusion is perhaps a *better* example because it is not a total order.

Definition 2.3 (**Total Order**) A total order on a set A is a partial order R such that for all $a, b \in A$, either aRb is true, or bRa is true (or both). ■

$\leq$ is a total order on $\mathbb{R}$, but $\subseteq$ is not a total order. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then $A \not\subseteq B$ and $B \not\subseteq A$, hence $\subseteq$ is not a total order. It is still a partial order. There is a special type of total order called *well-orderings.*

Definition 2.4 (**Well-Ordering**) A well-ordering on a set A is a total order $\leq$ on A such that for every non-empty subset $B \subseteq A$ there is a least element $a \in B$. That is, an element $a \in B$ such that for all other $b \in B$ it is true that $a \leq b$. ■

The natural numbers have this property. The standard ordering on the real numbers does *not* have this property. Let $A = (0, 1)$. There is no least element. If you pick a number $0 < r < 1$, I can pick $\frac{r}{2}$ and obtain a smaller number. It may seem impossible to order the real numbers in a way that gives a well-ordering, but the axiom of choice says it is possible.

Theorem 2.8. *If A is a set, then there exists a well-ordering $\leq$ on A.*

We will talk more about the well-ordering theorem when we discuss the *order topology* and the *long line.*