Point-Set Topology: Lecture 7

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July 3, 2023

1 Metric Topology

Definition 1.1 (**Metric Topology**) The metric topology on a metric space $(X,\,d)$ is the set $\tau_d \subseteq \mathcal{P}(X)$ such that for all $\mathcal{U}$, $\mathcal{U} \in \tau_d$ if and only if $\mathcal{U}$ is open in $(X,\,d)$. That is, τ_d is the set of all open subsets of $(X,\,d)$. ■

We have seen in previous theorems that the metric topology τ_d of a metric space $(X,\,d)$ has several properties. First, $\emptyset \in \tau_d$ and $X \in \tau_d$. That is, the empty set is open and the whole space is open. Secondly, given any subset $\mathcal{O} \subseteq \tau_d$, the union $\bigcup \mathcal{O}$ is an element of τ_d. That is, the arbitrary union of open sets is open. Lastly, if $\mathcal{U}, \mathcal{V} \in \tau_d$, then $\mathcal{U} \cap \mathcal{V} \in \tau_d$. That is, the finite intersection of open sets is open. We will take these properties and use them to define a topological space. A topological space is a set X and a subset $\tau \subseteq \mathcal{P}(X)$ with the four properties mentioned previously. This will be made clear in later lectures, for now we want to discuss which properties of a metric space are *topological* and which properties are geometric, or metric properties.

Definition 1.2 (**Topologically Equivalent Metrics**) Topologically equivalent metrics on a set X are metrics d_0 and d_1 such that their respective metric topologies τ_{d_0} and τ_{d_1} are equal, $\tau_{d_0} = \tau_{d_1}$. ■

To provide examples of equivalent metrics, it is best to use the following theorem.

Theorem 1.1. *If X is a set, and d_0 and d_1 are metrics on X, then d_0 and d_1 are topologically equivalent if and only if for all $x \in X$ and $r > 0$, there is an $r_0 > 0$ and an $r_1 > 0$ such that $B_{r_0}^{(X,\,d_0)}(x) \subseteq B_r^{(X,\,d_1)}(x)$ and $B_{r_1}^{(X,\,d_1)}(x) \subseteq B_r^{(X,\,d_0)}(x)$. That is, the open balls can be nested inside of each other.*

Proof. If $\tau_{d_0} = \tau_{d_1}$, then $B_r^{(X,\,d_0)}(x)$ is open in τ_{d_1} meaning there is an $r_1 > 0$ such that $B_{r_1}^{(X,\,d_1)}(x) \subseteq B_r^{(X,\,d_0)}(x)$. Similarly, if $\tau_{d_0} = \tau_{d_1}$, then $B_r^{(X,\,d_1)}(x)$ is open in τ_{d_0} meaning there is an $r_0 > 0$ such that $B_{r_0}^{(X,\,d_0)}(x) \subseteq B_r^{(X,\,d_0)}(x)$. In the other direction, suppose τ_{d_0} and τ_{d_1} are such that open balls can be nested. Let $\mathcal{V} \in \tau_{d_1}$. For all $x \in \mathcal{V}$, since $\mathcal{V}$ is open, there is an $r > 0$ such that $B_r^{(X,\,d_1)}(x) \subseteq \mathcal{V}$. But then there is an $r_0 > 0$ such that $B_{r_0}^{(X,\,d_0)}(x) \subseteq B_r^{(X,\,d_1)}(x)$, and hence $B_{r_0}^{(X,\,d_0)}(x) \subseteq \mathcal{V}$, so $\mathcal{V} \in \tau_{d_0}$. Similarly, if $\mathcal{U} \in \tau_{d_0}$, then for all $x \in \mathcal{U}$, since $\mathcal{U}$ is open, there is an $r > 0$ such that $B_r^{(X,\,d_0)}(x) \subseteq \mathcal{U}$. But then there is

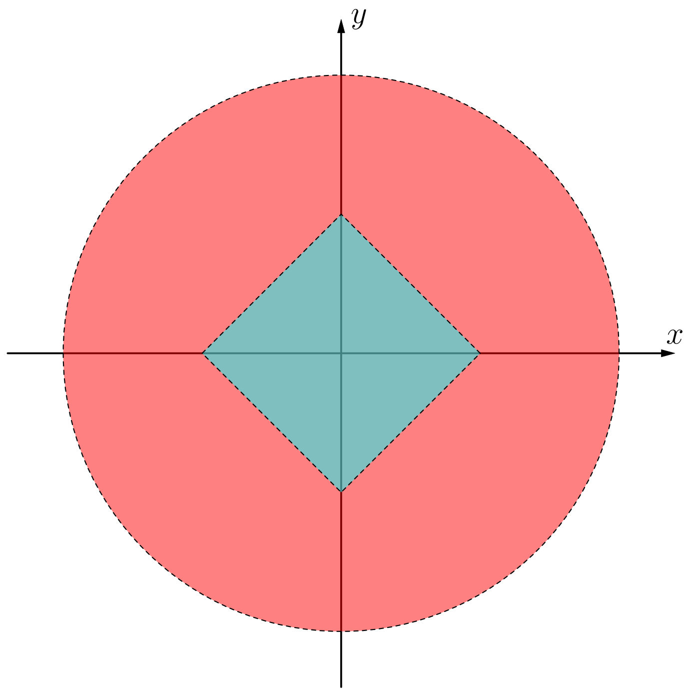

Figure 1: Manhattan Open Sets Nested in Euclidean Open Sets

an $r_1 > 0$ such that $B_{r_1}^{(X,\, d_1)}(x) \subseteq B_r^{(X,\, d_0)}(x)$, and hence $B_{r_1}^{(X,\, d_1)}(x) \subseteq \mathcal{U}$. That is, $\mathcal{U} \in \tau_{d_1}$. Therefore, $\tau_{d_0} = \tau_{d_1}$. □

Example 1.1 Let $(\mathbb{R}^2,\, d_E)$ be the Euclidean metric space on the plane, and $(\mathbb{R}^2,\, d_M)$ be the Manhattan metric space. Open balls in the Euclidean metric are open disks and open balls in the Manhattan metric are open diamonds. We can nest one inside of the other, showing that the Euclidean and Manhattan metrics are topologically equivalent. See Figs. 1 and 2. ■

Example 1.2 Let $(\mathbb{R}^2,\, d_E)$ be the Euclidean plane and $(\mathbb{R}^2,\, d_{\max})$ be the maximum metric space on the plane (the chess board metric). Open balls in the Euclidean plane are open disks and open balls in the max metric are open squares. We can nest one inside the other, meaning the Euclidean metric and the maximum metric are equivalent on $\mathbb{R}^2$ (See Figs. 3 and 4). ■

Example 1.3 The Euclidean metric and the Paris metric on $\mathbb{R}^2$ are not equivalent. Given a point $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{0}$, choose $r = ||\mathbf{x}||_2/2$, where $||\mathbf{x}||_2$ is the standard Euclidean length of the vector $\mathbf{x}$. The open ball centered at $\mathbf{x}$ with radius r is an open line segment going from $\mathbf{x}-(r,\, r)$ to $\mathbf{x}+(r,\, r)$. Open line segments are not open in the Euclidean metric, so the Paris metric is topologically different than the Euclidean metric. ■

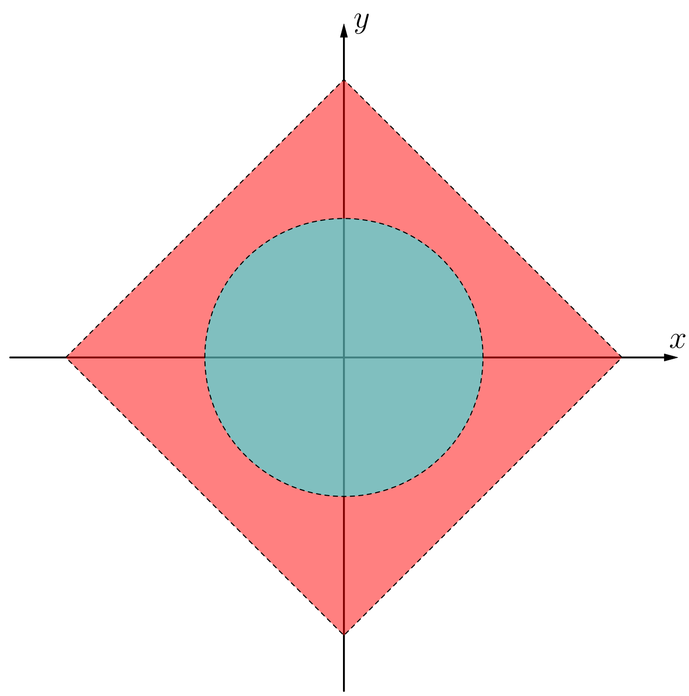

Figure 2: Euclidean Open Sets Nested in Manhattan Open Sets

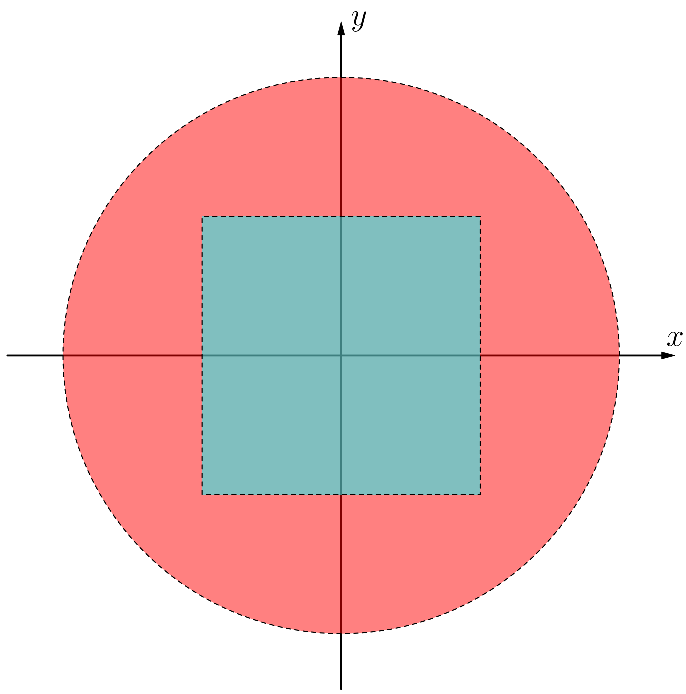

Figure 3: Max Open Sets Nested in Euclidean Open Sets

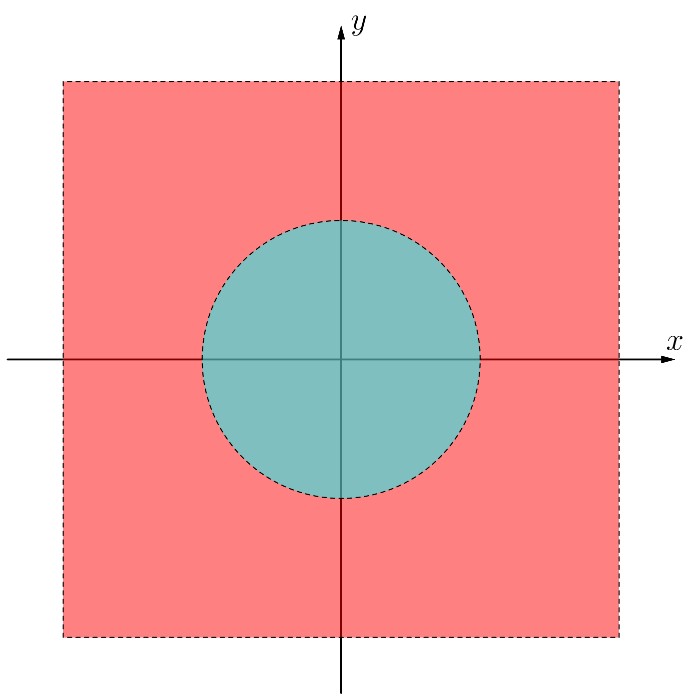

Figure 4: Euclidean Open Sets Nested in Max Open Sets

Example 1.4 The London metric and the Paris metric are not topologically equivalent. Given $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{0}$, choose $r = ||\mathbf{x}||_2/2$. The open ball of radius r centered about $\mathbf{x}$ in the Paris metric, as described before, is an open line segment in the plane. The open ball centered about $\mathbf{x}$ of radius r in the London metric is just the singleton $\{\,\mathbf{x}\,\}$. To see this, for all other points $\mathbf{y} \neq \mathbf{x}$, the distance in the London metric is:

$$d_L(\mathbf{x},\,\mathbf{y}) = ||\mathbf{x}||_2 + ||\mathbf{y}||_2 = 2r + ||\mathbf{y}||_2 > r \tag{1}$$

Meaning $\mathbf{y}$ is not in the ball of radius r centered about $\mathbf{x}$ in the London metric. So, the ball of radius r centered about $\mathbf{x}$ is just $\{\,\mathbf{x}\,\}$. Single points are not open in the Paris metric, showing the two metrics are not topologically equivalent. ■

Example 1.5 The London metric is not topologically equivalent to the discrete metric. It is true that for every point $\mathbf{x} \neq \mathbf{0}$, the point $\{\,\mathbf{x}\,\}$ is open in the London metric, which certainly seems similar to the discrete metric, but the set $\{\,\mathbf{0}\,\}$ is not open. An open ball about $\mathbf{0}$ in the London metric is an open disk, so $\{\,\mathbf{0}\,\}$ is not open. However, $\{\,\mathbf{0}\,\}$ is open in the discrete metric. ■

Topological properties are those that are detected by the topologies of the metric space. Convergence, continuity, open and closed, are all notions that are topological. As we will see repeatedly throughout the course, *homeomorphisms* are functions that preserve topological properties.

Definition 1.3 (**Homeomorphism**) A homeomorphism from a metric space $(X,\,d_X)$ to a metric space $(Y,\,d_Y)$ is a bijective continuous function $f : X \to Y$ such that f^{-1} is continuous. ■

Global isometries are functions that preserve all metric properties. Global isometries are, in particular, homeomorphisms.

Theorem 1.2. *If $(X,\,d_X)$ and $(Y,\,d_Y)$ are metric spaces, and if $f : X \to Y$ is a global isometry, then f is a homeomorphism.*

Proof. We have proven that isometries are continuous. All we need to do now is prove that if $f : X \to Y$ is a global isometry, then f^{-1} is also an isometry. Let $y_0, y_1 \in Y$. Since f is a global isometry, it is bijective, and hence there are $x_0, x_1 \in X$ with $f(x_0) = y_0$ and $f(x_1) = y_1$. But then, since f is an isometry, we have:

$$\begin{aligned} d_Y(y_0,\,y_1) &= d_Y\big(f(x_0),\,f(x_1)\big) && (2)\\ &= d_X(x_0,\,x_1) && (3)\\ &= d_X\big(f^{-1}(x_0),\,f^{-1}(x_1)\big) && (4) \end{aligned}$$

and hence f^{-1} is an isometry. But then f and f^{-1} are continuous, so f is a homeomorphism. □

Similar to how not every continuous function is an isometry, not every homeomorphism is a global isometry. Homeomorphism is a weaker notion, but also far more general and with more applications. Think about the real line. The only isometries are translations $f(x) = x + a$, reflections $f(x) = -x$, and glide reflections, $f(x) = -x + a$. Most of the functions used in calculus and physics are not isometries, but are usually continuous.

2 Completeness

Definition 2.1 (**Cauchy Sequences**) A Cauchy sequence in a metric space $(X,\, d)$ is a sequence $a : \mathbb{N} \to X$ such that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that given $m, n \in \mathbb{N}$ with $m > N$ and $n > N$, it is true that $d(a_n,\, a_m) < \varepsilon$. ■

Cauchy sequences are sequences where the points a_n start to get closer and closer together as n increases. Convergent sequences are, in particular, Cauchy sequences.

Theorem 2.1. *If $(X,\, d)$ is a metric space, and if $a : \mathbb{N} \to X$ is a convergent sequence, then a is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$. Since $a : \mathbb{N} \to X$ is convergent, there is an $x \in X$ with $a_n \to x$. But then there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$ it is true that $d(a_n,\, x) < \frac{\varepsilon}{2}$. But then for $m > N$ and $n > N$ we have:

$$d(a_m,\, a_n) \leq d(a_m,\, x) + d(a_n,\, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{5}$$

and therefore a is a Cauchy sequence. □

Since the points a_n are getting closer and closer together it is natural to ask if the converse of this theorem is true as well. That is, if $a : \mathbb{N} \to X$ is a Cauchy sequence in a metric space $(X,\, d)$, is a also a convergent sequence?

Example 2.1 Define $a : \mathbb{N} \to \mathbb{Q}$ by $a_0 = 1$, $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, and a_n is the first n decimals of $\sqrt{2}$. This is a Cauchy sequence, given $m < n$, $d(a_m,\, a_n)$ is less than 10^{-m}, which can be made arbitrarily small. It does not converge in $\mathbb{Q}$. The *limit* we want to say this converges to is $\sqrt{2}$, but $\sqrt{2}$ is not a rational number, so in reality there is no limit of this sequence. ■

The problem with $\mathbb{Q}$ is it has a lot of holes, these are the irrational numbers. If we fill in those holes we get the real numbers. This idea gives rise to the notion of *complete* metric spaces.

Definition 2.2 (**Complete Metric Space**) A complete metric space is a metric space $(X,\, d)$ such that for every Cauchy sequence $a : \mathbb{N} \to X$, it is true that a is a convergent sequence. ■

The real numbers are complete, with the usual metric $d(x,\, y) = |x - y|$. Given a bounded set $A \subseteq \mathbb{R}$, meaning there is an $M \in \mathbb{R}$ such that for all $x \in A$ it is true that $|x| < M$, the real numbers have the property that A has a *least upper bound* and a *greatest lower bound*. That is, numbers r and s such that r is a lower bound, all $x \in A$ are such that $r \leq x$, and s is an upper bound, all $x \in A$ are such that $x \leq s$, but moreover r is the *largest* possible lower bound, and s is the *smallest* possible upper bound. The rationals do not have this property. Given the set $A = \{\, x \in \mathbb{Q} \mid x^2 < 2 \,\}$, there is no least upper bound. If you give me a rational number that is an upper bound for A, I can find a smaller rational number that is also an upper bound. The *least* upper bound of this set is $\sqrt{2}$, but again, this is not a rational number. This least upper bound property can be used to show that the real numbers are complete. First, a Cauchy sequence is bounded. Given $a : \mathbb{N} \to \mathbb{R}$ a Cauchy sequence, the points a_n are getting really close together, so it would be impossible for the sequence to diverge off to infinity. Using this we consider the set of all real numbers r where there are infinitely many a_n less than r. Since the a_n are bounded, this set is non-empty and bounded above, so there is a least upper bound. Using a bit of work, you can then show that this Cauchy sequence must converge to this least upper bound, and *viola*, you have proven that the real numbers are complete.

Completeness motivates a more topological notion, *compactness*. We'll first introduce a few more metric notions before heading into this topic.

Definition 2.3 (**Bounded Metric Space**) A bounded metric space is a metric space $(X,\,d)$ such that there is an $M > 0$ such that for all $x, y \in X$ it is true that $d(x,\,y) < M$. ■

Example 2.2 The real line is not bounded with the standard metric. Given any $M > 0$, choose $x = 0$ and $y = M + 1$. Then $d(x,\,y) = |x - y| = M + 1$ which is greater than M. ■

Example 2.3 The real line with the arctan metric $d(x,\,y) = |\text{atan}(x) - \text{atan}(y)|$, is bounded with bound $M = \pi$. ■

Example 2.4 The circle $\mathbb{S}^1$ with the subspace metric from $\mathbb{R}^2$ is bounded, any number $M > 2$ suffices as a bound. ■

Is boundedness a topological property? That is, if d_0 and d_1 are equivalent metrics on X, and if d_0 is unbounded, is d_1 also unbounded?

Theorem 2.2. *If $(X,\,d)$ is a metric space, then there exists a topologically equivalent metric ρ such that $(X,\,\rho)$ is a bounded metric space.*

Proof. Let $\rho : X \times X \to \mathbb{R}$ be defined by:

$$\rho(x,\,y) = \frac{d(x,\,y)}{1 + d(x,\,y)} \tag{6}$$

ρ is a metric on X. It is positive-definite since the denominator is always positive and the numerator is positive-definite. It is symmetric since:

$$\rho(y,\,x) = \frac{d(y,\,x)}{1+d(y,\,x)} \tag{7}$$
$$= \frac{d(x,\,y)}{1+d(x,\,y)} \tag{8}$$
$$= \rho(x,\,y) \tag{9}$$

Lastly, the triangle inequality. There are two cases. Case 1, $d(x,\,y) \leq d(x,\,z)$ and $d(y,\,z) \leq d(x,\,z)$. We get:

$$\rho(x,\,z) = \frac{d(x,\,z)}{1+d(x,\,z)} \tag{10}$$
$$\leq \frac{d(x,\,y)+d(y,\,z)}{1+d(x,\,z)} \tag{11}$$
$$= \frac{d(x,\,y)}{1+d(x,\,z)} + \frac{d(y,\,z)}{1+d(x,\,z)} \tag{12}$$
$$\leq \frac{d(x,\,y)}{1+d(x,\,y)} + \frac{d(y,\,z)}{1+d(y,\,z)} \tag{13}$$
$$= \rho(x,\,y)+\rho(y,\,z) \tag{14}$$

Case 2, $d(x,\,z) \leq d(x,\,y)$ or $d(x,\,z) \leq d(y,\,z)$. Since the function $f(x) = \frac{x}{1+x}$ is strictly increasing on the set $[0,\infty)$ we get $\rho(x,\,z) \leq \rho(x,\,y)$ or $\rho(x,\,z) \leq \rho(y,\,z)$, and hence $\rho(x,\,z) \leq \rho(x,\,y) + \rho(y,\,z)$. This metric is topologically equivalent. Given $r > 0$, let $r_d = r$. Then if $y \in B_{r_d}^{(X,\,d)}(x)$, we have:

$$\rho(x,\,y) = \frac{d(x,\,y)}{1+d(x,\,y)} < d(x,\,y) < r_d = r \tag{15}$$

and hence $y \in B_r^{(X,\,\rho)}(x)$. That is, $B_{r_d}^{(X,\,d)}(x) \subseteq B_r^{(X,\,\rho)}(x)$. If $r < 1$, let $r_\rho = \frac{r}{1-r}$. If $y \in B_{r_\rho}^{(X,\,\rho)}(x)$, then (since $\frac{x}{1-x}$ is strictly increasing for $0 < x < 1$):

$$d(x,\,y) = \frac{\rho(x,\,y)}{1-\rho(x,\,y)} < \frac{r_\rho}{1-r_\rho} = r \tag{16}$$

so $y \in B_r^{(X,\,d)}(x)$, and hence $B_{r_\rho}^{(X,\,\rho)}(x) \subseteq B_r^{(X,\,d)}(x)$. If $r \geq 1$, let $r' = \frac{1}{2}$ and $r_\rho = \frac{r'}{1-r'}$. Since $r' < r$, $B_{r'}^{(X,\,d)}(x) \subseteq B_r^{(X,\,d)}(x)$ and hence $B_{r_\rho}^{(X,\,\rho)} \subseteq B_r^{(X,\,d)}(x)$. By the theorem at the start of these notes, $(X,\,d)$ and $(X,\,\rho)$ are topologically equivalent. Moreover, since $\frac{x}{1+x}$ is bounded on $[0,\infty)$, $(X,\,\rho)$ is a bounded metric space. □

Definition 2.4 (**Totally Bounded Metric Space**) A totally bounded metric space is a metric space $(X,\,d)$ such that for all $\varepsilon > 0$ there are finitely many points $a_0,\,\dots,\,a_N$ such that:

$$X = \bigcup_{n=0}^{N} B_\varepsilon^{(X,\,d)}(a_n) \tag{17}$$

That is, the set of ε balls centered about the points a_n completely cover the metric space. ■

Theorem 2.3. *If $(X,\,d)$ is a totally bounded metric space, then $(X,\,d)$ is a bounded metric space.*

Proof. Let $(X,\,d)$ be totally bounded, and let $\varepsilon = 1$. There exists finitely many points $a_0,\,\dots,\,a_N$ such that

$$X = \bigcup_{n=0}^{N} B_{\varepsilon}^{(X,\,d)}(a_n) \tag{18}$$

Let r be the maximum value of $d(a_n,\,a_m)$ for all $0 \leq m, n \leq N$, and let $M = r + 2$. Let $x, y \in X$ be arbitrary. There are two points a_m, a_n such that $x \in B_1^{(X,\,d)}(a_m)$ and $y \in B_1^{(X,\,d)}(a_n)$ since these ε balls cover the entirety of X. But then:

$$d(x,\,y) \leq d(x,\,a_m) + d(a_m,\,a_n) + d(a_n,\,y) < 1 + r + 1 = M \tag{19}$$

so M is a bound for $(X,\,d)$. □

The converse need not be true.

Example 2.5 Equip $\mathbb{R}$ with the discrete metric:

$$d(x,\,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases} \tag{20}$$

The metric space $(\mathbb{R},\,d)$ is bounded by 2. It is not totally bounded. Given $\varepsilon = \frac{1}{2}$, the only way to cover $\mathbb{R}$ with ε balls is by placing an ε ball about every real number $r \in \mathbb{R}$, so we can't possibly cover $\mathbb{R}$ with finitely many ε balls with the discrete metric, even though the space is bounded. ■

Theorem 2.4 (**Bolzano's Theorem**). *If $a : \mathbb{N} \to \mathbb{R}$ is a sequence, then there is monotone subsequence. That is, a subsequence a_k such that for all $n \in \mathbb{N}$ it is true that $a_{k_n} \leq a_{k_{n+1}}$ (monotone increasing), or such that for all $n \in \mathbb{N}$ it is true that $a_{k_n} \geq a_{k_{n+1}}$ (monotone decreasing).*

I'll give a sketch of proof via a picture. For simplicity, suppose $a : \mathbb{N} \to \mathbb{R}^+$ is a sequence of positive numbers. Place a flashlight infinitely far away at $-\infty$ on the x axis. Given the element $a_n \in \mathbb{R}$ of the sequence, draw a straight line from $(n,\,a_n)$ to $(n,\,0)$. These act as *walls*. With the light shining from behind, some walls will be lit and some will not. If there are infinitely many walls that recieve some light, then we must have a monotone subsequence. Simply go to the next wall that is lit up, and keep doing this to obtain your subsequence. If not, there is a *tallest* wall that keeps all the other walls in the shade. Start there and go to the next tallest wall, and then the next tallest wall, and so on, obtaining a monotone subsequence. See Fig. 5.

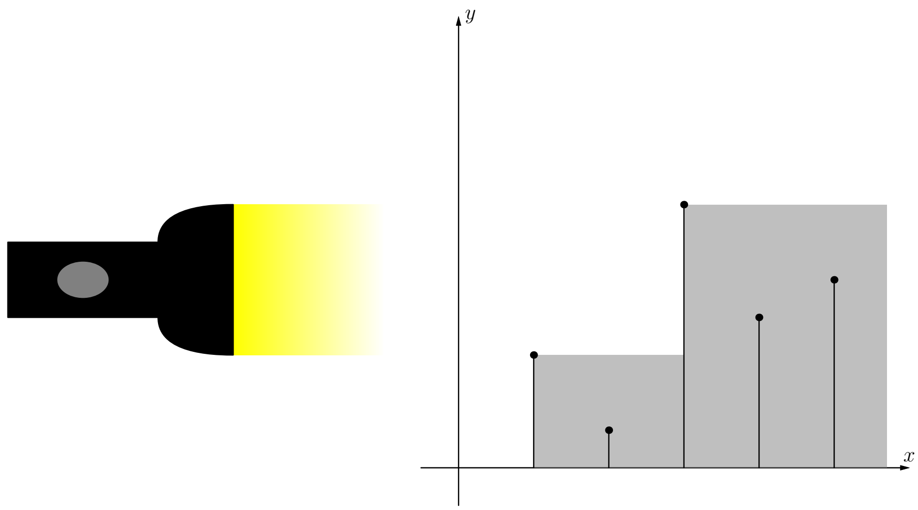

Figure 5: Sketch of Proof of Bolzano's Theorem

The full proof of this theorem belongs in a course on real analysis, but the idea of the proof is essentially the idea discussed above. We will use it to prove the *Bolzano-Weierstrass theorem*, a core theorem to real analysis that completely motivates the idea of compact metric spaces.

Theorem 2.5 (**Bolzano-Weierstrass Theorem**). *If $a:\mathbb{N}\rightarrow\mathbb{R}$ is a bounded sequence, then there is a convergence subsequence a_k.*

Proof. Let $a:\mathbb{N}\rightarrow\mathbb{R}$ be a bounded sequence. By Bolzano's theorem there is a monotone subsequence a_k. Suppose a_k is monotone increasing (the idea is symmetric if a_k is monotone decreasing). Since a is a bounded sequence, a_k is also a bounded sequence. Let $x\in\mathbb{R}$ be the least upper bound of this subsequence. Let $\varepsilon>0$. Since x is the least upper bound, $x-\varepsilon$ is not an upper bound of the sequence (otherwise x is not the *least* upper bound since $x-\varepsilon$ is smaller). But if $x-\varepsilon$ is not an upper bound, then there is an $N\in\mathbb{N}$ with $a_{k_N}>x-\varepsilon$. But a_k is monotone increasing, so for all $n>N$, $a_{k_n}\geq a_{k_N}$ and therefore $a_{k_n}>x-\varepsilon$. But also $a_{k_n}\leq x$ since x is the least upper bound. That is, for all $n>N$ we have $x-\varepsilon<a_{k_n}\leq x$. Therefore, for $n>N$, we have $|x-a_{k_n}|<\varepsilon$, and hence a_{k_n} converges to x. □

We take this idea and use it to define compactness.

Definition 2.5 (**Compact Metric Space**) A compact metric space is a metric space $(X,\,d)$ such that for every sequence $a:\mathbb{N}\rightarrow X$ there is a convergent subsequence a_k. ■