Point-Set Topology: Lecture 12

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1 Closure, Interior, and Boundary

In previous lectures we used large collections of topologies to generate a new one. In particular, we took a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$, and looked at the set T of all topologies τ on X such that $\mathcal{B} \subseteq \tau$. This set T is non-empty since $\mathcal{P}(X) \in T$. We then created a new topology via the intersection $\bigcap T$. This is the *generated* topology. We now use a similar idea, but instead of collections of topologies, we look at collections of open and closed sets. We've seen some laws about open sets, these are the rules dictated by the definition of a topology. Using the De Morgan law's we get similar statements about closed sets.

Theorem 1.1. *If (X, τ) is a topological space, then $\emptyset$ is closed.*

Proof. Since X is open and $\emptyset = X \setminus X$, $\emptyset$ is closed. □

Theorem 1.2. *If (X, τ) is a topological space, then X is closed.*

Proof. Since $\emptyset$ is open and $X = X \setminus \emptyset$, X is closed. □

Theorem 1.3. *If (X, τ) is a topological space, and if $\mathcal{C}, \mathcal{D} \subseteq X$ are closed, then $\mathcal{C} \cup \mathcal{D}$ is closed.*

Proof. Since $\mathcal{C}$ and $\mathcal{D}$ are closed, $X \setminus \mathcal{C}$ and $X \setminus \mathcal{D}$ are open. But then:

$$X \setminus (\mathcal{C} \cup \mathcal{D}) = (X \setminus \mathcal{C}) \cap (X \setminus \mathcal{D}) \tag{1}$$

which is the intersection of two open sets, which is therefore open, so $X \setminus (\mathcal{C} \cup \mathcal{D})$ is open. But then $\mathcal{C} \cup \mathcal{D}$ is closed. □

Theorem 1.4. *If (X, τ) is a topological space, and if $\mathcal{O} \subseteq \mathcal{P}(X)$ is such that for all $\mathcal{C} \in \mathcal{O}$ it is true that $\mathcal{C}$ is closed, then $\bigcap \mathcal{O}$ is closed.*

Proof. If $\mathcal{O}$ is empty, then $\bigcap \mathcal{O} = \emptyset$, which is closed. Otherwise we may write:

$$\bigcap \mathcal{O} = \bigcap_{\mathcal{C} \in \mathcal{O}} \mathcal{C} = \bigcap_{\mathcal{C} \in \mathcal{O}} \Big(X \setminus (X \setminus \mathcal{C}) \Big) = X \setminus \bigcup_{\mathcal{C} \in \mathcal{O}} (X \setminus \mathcal{C}) \tag{2}$$

Since all $\mathcal{C}$ are closed, $X \setminus \mathcal{C}$ is open, so this union is open, meaning $\bigcap \mathcal{O}$ is the complement of an open set and is therefore closed. □

Theorem 1.5. *If (X, τ) is a topological space, and if $\mathcal{O} \subseteq \mathcal{P}(X)$ is a finite set such that for all $\mathcal{C} \in \mathcal{O}$ it is true that $\mathcal{C}$ is closed, then $\bigcup \mathcal{O}$ is closed.*

Proof. We prove by induction. The base case is true by a previous theorem. Suppose the statement is true for all such $\mathcal{O}$ with n elements. Now, let $\mathcal{O}$ be a set of $n+1$ closed sets. That is, we may write $\mathcal{O} = \{\,\mathcal{C}_0, \ldots, \mathcal{C}_n\,\}$. Define $\mathcal{D}$ via:

$$\mathcal{D} = \bigcup_{k=0}^{n-1} \mathcal{C}_k \tag{3}$$

Then $\mathcal{D}$ is the union of n closed sets, and by the induction hypothesis it is closed. But then:

$$\bigcup \mathcal{O} = \bigcup_{k=0}^{n} \mathcal{C}_k = \mathcal{D} \cup \mathcal{C}_n \tag{4}$$

which is the union of two closed sets, which is closed. Hence, by induction, $\bigcup \mathcal{O}$ is closed for any finite collection of closed sets. □

We use the intersection property to define *closure*. Given *any* subset $A \subseteq X$ in a topological space (X, τ) there is at least one closed set containing A since $A \subseteq X$ and X is closed. The *closure* of A is the *smallest* closed set containing A. We can be very precise about this.

Definition 1.1 (**Closure of a Set**) The closure of a subset $A \subseteq X$ in a topological space (X, τ) is the set $\mathrm{Cl}_\tau(A)$ defined by:

$$\mathrm{Cl}_\tau(A) = \bigcap \{\,\mathcal{C} \subseteq X \mid \mathcal{C} \text{ is closed and } A \subseteq \mathcal{C}\,\} \tag{5}$$

That is, the *smallest* closed set containing A. ■

Theorem 1.6. *If (X, τ) is a topological space and $A \subseteq X$, then $A \subseteq Cl_\tau(A)$.*

Proof. Let $\mathcal{O}$ be the set of all closed sets containing A. This set is non-empty since $X \in \mathcal{O}$. Given any element $\mathcal{C} \in \mathcal{O}$ we have $A \subseteq \mathcal{C}$ by definition. Hence, $A \subseteq \bigcap \mathcal{O}$. But $\mathrm{Cl}_\tau(A) = \bigcap \mathcal{O}$, completing the proof. □

Theorem 1.7. *If (X, τ) is a topological space, and if $A \subseteq X$, then $Cl_\tau(A)$ is closed.*

Proof. Since $\mathrm{Cl}_\tau(A)$ is the intersection of closed sets, it is closed. □

Theorem 1.8. *If (X, τ) is a topological space, then $\mathcal{C} \subseteq X$ is closed if and only if $Cl_\tau(\mathcal{C}) = \mathcal{C}$.*

Proof. If $\mathcal{C} = \mathrm{Cl}_\tau(\mathcal{C})$, then $\mathcal{C}$ is closed since $\mathrm{Cl}_\tau(\mathcal{C})$ is closed. In the other direction, if $\mathcal{C}$ is closed, then $\mathcal{C}$ is a closed set that contains $\mathcal{C}$ since $\mathcal{C} \subseteq \mathcal{C}$. But then $\mathrm{Cl}_\tau(\mathcal{C}) \subseteq \mathcal{C}$. But $\mathcal{C} \subseteq \mathrm{Cl}_\tau(\mathcal{C})$ is also true, so $\mathcal{C} = \mathrm{Cl}_\tau(\mathcal{C})$. □

Theorem 1.9. *If (X, τ) is a topological space, and if $A \subseteq X$, then:*

$$Cl_\tau\big(Cl_\tau(A)\big) = Cl_\tau(A) \tag{6}$$

Proof. Since $\mathrm{Cl}_\tau(A)$ is closed, we have that $\mathrm{Cl}_\tau\big(\mathrm{Cl}_\tau(A)\big) = \mathrm{Cl}_\tau(A)$ by the previous theorem. $\square$

Theorem 1.10. *If $(X,\, \tau)$ is a topological space, if $A, B \subseteq X$, then:*

$$Cl_\tau(A \cup B) = Cl_\tau(A) \cup Cl_\tau(B) \tag{7}$$

Proof. Since $\mathrm{Cl}_\tau(A)$ and $\mathrm{Cl}_\tau(B)$ are closed, and since $A \subseteq \mathrm{Cl}_\tau(A)$ and $B \subseteq \mathrm{Cl}_\tau(A)$, we have that $\mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B)$ is a closed set (since it is the union of two closed sets) that contains $A \cup B$. Hence $\mathrm{Cl}_\tau(A \cup B) \subseteq \mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B)$. But a closed set that contains $A \cup B$ is also a closed set that contains A, and a closed set that contains $A \cup B$ is also a closed set that contains B, so $\mathrm{Cl}_\tau(A) \subseteq \mathrm{Cl}_\tau(A \cup B)$ and $\mathrm{Cl}_\tau(B) \subseteq \mathrm{Cl}_\tau(A \cup B)$. Thus, $\mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B) \subseteq \mathrm{Cl}_\tau(A \cup B)$. Therefore, $\mathrm{Cl}_\tau(A \cup B) = \mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B)$. $\square$

Theorem 1.11. *If $(X,\, \tau)$ is a topological space, then $Cl_\tau(\emptyset) = \emptyset$.*

Proof. This follows since $\emptyset$ is closed, and the closure of a closed set is itself. $\square$

Theorem 1.12. *If $(X,\, \tau)$ is a topological space, then $Cl_\tau(X) = X$.*

Proof. This also follows since X is closed. $\square$

Theorem 1.13. *If $(X,\, \tau)$ is a topological space, if $A, B \subseteq X$, and if $A \subseteq B$, then $Cl_\tau(A) \subseteq Cl_\tau(B)$.*

Proof. Let T_A be the set of all closed subsets of X that contain A, and similarly define T_B. Since $A \subseteq B$, if $\mathcal{C} \subseteq X$ is a closed subset such that $B \subseteq \mathcal{C}$, since inclusion is transitive we have $A \subseteq \mathcal{C}$. That is $\mathcal{C} \in T_B$ implies $\mathcal{C} \in T_A$, and hence $T_B \subseteq T_A$. Intersections are order reversing, and hence $\bigcap T_A \subseteq \bigcap T_B$, meaning $\mathrm{Cl}_\tau(A) \subseteq \mathrm{Cl}_\tau(B)$. $\square$

Example 1.1 Take $\mathbb{R}$ with the standard topology. Let $\mathbb{Q}$ be the set of all rational numbers. The closure of $\mathbb{Q}$ is all of $\mathbb{R}$. Every real number can be written as a limit point of $\mathbb{Q}$ since we may approximate any $x \in \mathbb{R}$ with a convergent sequence of rational numbers. Because of this $\mathrm{Cl}_{\tau_\mathbb{R}}(\mathbb{Q}) = \mathbb{R}$. $\blacksquare$

Example 1.2 In the real line $\mathbb{R}$ with the standard topology $\tau_\mathbb{R}$, given $a, b \in \mathbb{R}$ with $a < b$, the closure of $(a,\, b)$ is the set $[a,\, b]$. In a metric space you can obtain the closure of a set by adding all of the limit points (points that can be approximated via sequences) of the set. Since a and b are limit points of $(a,\, b)$, we see that $\mathrm{Cl}_{\tau_\mathbb{R}}\big((a,\, b)\big) = [a,\, b]$. $\blacksquare$

The interior of a set uses similar ideas, but using open sets and unions.

Definition 1.2 (**Interior of a Set**) The interior of a subset $A \subseteq X$ in a topological space $(X,\, \tau)$ is the set $\mathrm{Int}_\tau(A)$ defined by:

$$\mathrm{Int}_\tau(A) = \bigcup \{\, \mathcal{U} \in \tau \mid \mathcal{U} \subseteq A \,\} \tag{8}$$

That is, the *largest* open set that is contained inside of A. $\blacksquare$

Theorem 1.14. *If (X, τ) is a topological space and $A \subseteq X$, then $Int_\tau(A) \subseteq A$.*

Proof. Since $\text{Int}_\tau(A)$ is the union over open sets that are contained in A, the union is contained in A, meaning $\text{Int}_\tau(A) \subseteq A$. □

Theorem 1.15. *If (X, τ) is a topological space and $A \subseteq X$, then $Int_\tau(A)$ is open.*

Proof. Since $\text{Int}_\tau(A)$ is the union of open sets, it is open. □

Theorem 1.16. *If (X, τ) is a topological space, and if $\mathcal{U} \subseteq X$, then $\mathcal{U} \in \tau$ if and only if $Int_\tau(\mathcal{U}) = \mathcal{U}$.*

Proof. If $\mathcal{U} = \text{Int}_\tau(\mathcal{U})$, then $\mathcal{U}$ is equal to an open set, and so is open. In the other direction, if $\mathcal{U}$ is open, then $\mathcal{U}$ is an open set that is contained in $\mathcal{U}$ since $\mathcal{U} \subseteq \mathcal{U}$. But then $\mathcal{U} \subseteq \text{Int}_\tau(\mathcal{U})$. But $\text{Int}_\tau(\mathcal{U}) \subseteq \mathcal{U}$, so $\mathcal{U} = \text{Int}_\tau(\mathcal{U})$. □

Theorem 1.17. *If (X, τ) is a topological space, and if $A \subseteq X$, then:*

$$Int_\tau\big(Int_\tau(A)\big) = Int_\tau(A) \tag{9}$$

Proof. Since $\text{Int}_\tau(A)$ is open, we have that $\text{Int}_\tau\big(\text{Int}_\tau(A)\big) = \text{Int}_\tau(A)$ by the previous theorem. □

Theorem 1.18. *If (X, τ) is a topological space, and if $A, B \subseteq X$, then:*

$$Int_\tau(A \cap B) = Int_\tau(A) \cap Int_\tau(B) \tag{10}$$

Proof. Since $\text{Int}_\tau(A \cap B)$ is an open set that is contained inside of A, we have $\text{Int}_\tau(A \cap B) \subseteq \text{Int}_\tau(A)$. But $\text{Int}_\tau(A \cap B)$ is also an open set contained inside of B, so $\text{Int}_\tau(A \cap B) \subseteq \text{Int}_\tau(B)$. But then $\text{Int}_\tau(A \cap B) \subseteq \text{Int}_\tau(A) \cap \text{Int}_\tau(B)$. Since $\text{Int}_\tau(A)$ and $\text{Int}_\tau(B)$ are open, $\text{Int}_\tau(A) \cap \text{Int}_\tau(B)$ is open. But this is an open set that is contains inside of $A \cap B$, meaning $\text{Int}_\tau(A) \cap \text{Int}_\tau(B) \subseteq \text{Int}_\tau(A \cap B)$. Hence, $\text{Int}_\tau(A \cap B) = \text{Int}_\tau(A) \cap \text{Int}_\tau(B)$. □

Theorem 1.19. *If (X, τ) is a topological space, then $Int_\tau(\emptyset) = \emptyset$.*

Proof. Since $\emptyset$ is open, it is equal to its interior. □

Theorem 1.20. *If (X, τ) is a topological space, then $Int_\tau(X) = X$.*

Proof. This also follows from the fact that X is open. □

Theorem 1.21. *If (X, τ) is a topological space, if $A, B \subseteq X$, and if $A \subseteq B$, then $Int_\tau(A) \subseteq Int_\tau(B)$.*

Proof. If $x \in \text{Int}_\tau(A)$, then there is an open set $\mathcal{U} \subseteq A$ such that $x \in \mathcal{U}$. But since $A \subseteq B$ we have $\mathcal{U} \subseteq B$, and since $\mathcal{U}$ is open it is true that $\mathcal{U} \subseteq \text{Int}_\tau(B)$. Therefore $x \in \text{Int}_\tau(B)$, so $\text{Int}_\tau(A) \subseteq \text{Int}_\tau(B)$. □

Example 1.3 Let $A = \mathbb{Q}$ as a subset of the standard topology on $\mathbb{R}$. The interior $\text{Int}_{\tau_{\mathbb{R}}}(\mathbb{Q})$ is *empty*. The only open subset $\mathcal{U} \subseteq \mathbb{Q}$ is the empty set. Given any $x \in \mathbb{Q}$ and any positive $\varepsilon > 0$ there are points $y \in \mathbb{R}$ such that y is irrational and $|x - y| < \varepsilon$. So there are no open balls centered about any rational points that contain only rational numbers, meaning $\text{Int}_{\tau_{\mathbb{R}}}(\mathbb{Q}) = \emptyset$. ■

Example 1.4 Given the standard topology on $\mathbb{R}$, $\tau_{\mathbb{R}}$, the interior of the closed interval $[a,\, b]$ with $a < b$ is the open interval $(a,\, b)$. This is the largest open subset of $[a,\, b]$. ■

Example 1.5 Let $X = [0,\, 1]$ and τ_X be the topology induced by the subspace metric. That is, given the standard metric $d(x,\, y) = |x - y|$ on $\mathbb{R}$, we create the subspace metric $d_X(x,\, y) = d(x,\, y)$ for all $x, y \in X$. This induces a topology on $[0,\, 1]$. What is the interior of $[0,\, 1]$ with respect to the topology τ_X? It is tempting to say the interior is $(0,\, 1)$, but this is **false**. Do not confuse the topology $\tau_{\mathbb{R}}$ with the topology τ_X. In $\mathbb{R}$, the interior of $[0,\, 1]$ is indeed $(0,\, 1)$. In τ_X the interior of $[0,\, 1]$ is $[0,\, 1]$. We are not considering $X = [0,\, 1]$ as a subset of the real line anymore, but rather as it's own topological space $(X,\, \tau_X)$. In this topological space the number 2 does not exist, nor does -1. The entire space X is open in τ_X, so $\text{Int}_{\tau_X}(X) = X$. ■

Definition 1.3 (**Topological Boundary**) The boundary of a subset $A \subseteq X$ in a topological space $(X,\, \tau)$ is the set $\partial_\tau(A) = \text{Cl}_\tau(A) \setminus \text{Int}_\tau(A)$. ■

Boundaries are always closed. This is because the set difference of an open set from a closed set is always closed.

Theorem 1.22. *If $(X,\, \tau)$ is a topological space, if $\mathcal{U} \in \tau$, and if $\mathcal{C}$ is closed, then $\mathcal{C} \setminus \mathcal{U}$ is closed.*

Proof. Since $\mathcal{U}, \mathcal{C} \subseteq X$, we can use the following fact from set theory. If A, B, and C are sets, and if $A, B \subseteq C$, then:

$$A \setminus B = A \cap (C \setminus B) \tag{11}$$

We have (with $A = \mathcal{C}$, $B = \mathcal{U}$, and $C = X$):

$$\mathcal{C} \setminus \mathcal{U} = \mathcal{C} \cap (X \setminus \mathcal{U}) \tag{12}$$

But $\mathcal{U}$ is open, so $X \setminus \mathcal{U}$ is closed. But $\mathcal{C}$ is closed, so this is the intersection of two closed sets, which is closed. Therefore, $\mathcal{C} \setminus \mathcal{U}$ is closed. □

Theorem 1.23. *If $(X,\, \tau)$ is a topological space, and if $A \subseteq X$, then $\partial_\tau(A)$ is closed.*

Proof. Applying the previous theorem, since $\text{Cl}_\tau(A)$ is closed and $\text{Int}_\tau(A)$ is open, $\partial_\tau(A) = \text{Cl}_\tau(A) \setminus \text{Int}_\tau(A)$ is closed. □

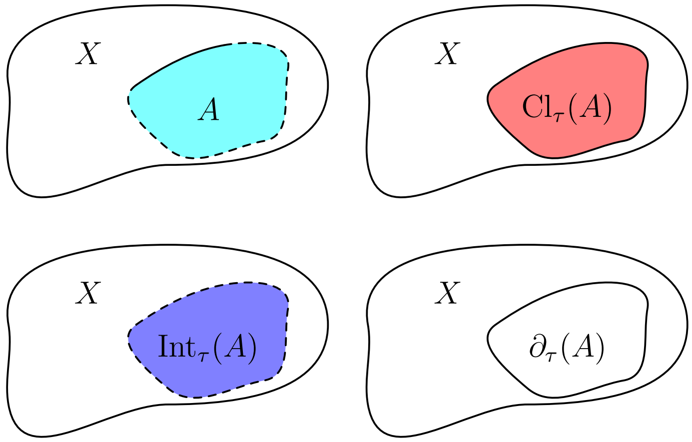

Figure 1: Interior, Closure, and Boundary

2 Sequences and Convergence

Convergence in a metric space required the metric, but we can alter the definition to only use open sets. In a metric space $(X,\, d)$, we said $a:\mathbb{N}\rightarrow X$ converges to $x\in X$, written $a_n\rightarrow x$, if for all $\varepsilon>0$ there is an $N\in\mathbb{N}$ such that $n\in\mathbb{N}$ and $n>N$ implies $d(x,\, a_n)<\varepsilon$. Worded differently, the sequence is *eventually* contained inside the ε ball centered at x for all $\varepsilon>0$. ε balls are, in particular, open sets, so we can say that $a_n\rightarrow x$ if for every open set $\mathcal{U}\subseteq X$ such that $x\in\mathcal{U}$, there is an $N\in\mathbb{N}$ such that $n\in\mathbb{N}$ and $n>N$ implies $a_n\in\mathcal{U}$. This final definition, which is equivalent to the metric one, relies only on open sets and can be phrased in a topological space.

Definition 2.1 (**Convergent Sequence in a Topological Space**) A convergent sequence in a topological space $(X,\, \tau)$ is a sequence $a:\mathbb{N}\rightarrow X$ such that there is an $x\in X$ such that for all $\mathcal{U}\in\tau$ with $x\in\mathcal{U}$ there is an $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ with $n>N$ it is true that $a_n\in\mathcal{U}$. We write $a_n\rightarrow x$. ■

The first theorem we proved in a metric space was that limits are unique, meaning we can say *the* limit of the sequence. This is not true in a general topological space. This is a very important distinction. In metric spaces we used sequences to define continuity. We can still use this in topological space. Given $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$, we can require a function $f:X\rightarrow Y$ to be such that if $a:\mathbb{N}\rightarrow X$ is a convergent sequence such that $a_n\rightarrow x$ with $x\in X$, then $f(a_n)$ is a convergent sequence in Y and $f(a_n)\rightarrow f(x)$. This type of function does indeed get a name, it's called a *sequentially continuous* function. It is inadequate to describe general continuity in a general topological space. The following examples should show why.

Example 2.1 Equip $\mathbb{R}$ with the indiscrete topology, $\tau=\{\,\emptyset,\,\mathbb{R}\,\}$. Let $a:\mathbb{N}\rightarrow\mathbb{R}$ be *any* sequence. Then for all $x\in\mathbb{R}$, the sequence a converges to x. Let's prove this. To show $a_n \to x$ we need to show that for all $\mathcal{U} \in \tau$ with $x \in \mathcal{U}$ there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $a_n \in \mathcal{U}$. But the only open sets in τ are $\emptyset$ and $\mathbb{R}$. So if $x \in \mathcal{U}$, then $\mathcal{U} \neq \emptyset$, so $\mathcal{U} = \mathbb{R}$. Pick $N = 0$. Then for all $n \in \mathbb{N}$ with $n > N$, since $a : \mathbb{N} \to \mathbb{R}$ is a sequence in $\mathbb{R}$, we have that $a_n \in \mathbb{R}$. This shows that $a_n \to x$ regardless of $x \in \mathbb{R}$. ■

Example 2.2 There's nothing special about $\mathbb{R}$ for the previous example, the set is just more concrete for visualization. If X is a set, and $\tau = \{\,\emptyset,\, X\,\}$ is the indiscrete topology on X, then given any sequence $a : \mathbb{N} \to X$, and any $x \in X$, it is true that $a_n \to x$. ■

Example 2.3 Let $X = \mathbb{N}$ and τ be the set of all $\mathbb{Z}_n$, $n \in \mathbb{N}$, together with $\mathbb{N}$. That is:

$$\tau = \{\,\mathbb{Z}_n \mid n \in \mathbb{N}\,\} \cup \{\,\mathbb{N}\,\} \tag{13}$$

$(\mathbb{N},\, \tau)$ is a topological space. Let $a : \mathbb{N} \to \mathbb{N}$ be the sequence:

$$a_n = \begin{cases} 1 & n \text{ is even} \\ 2 & n \text{ is odd} \end{cases} \tag{14}$$

Does a_n converge to 0? No, let $\mathcal{U} = \mathbb{Z}_1 = \{\,0\,\}$. Then a_n is never contained in the set $\mathcal{U}$, so a_n can not converge to 0. Does a_n converge to 1? Also no. Let $\mathcal{V} = \mathbb{Z}_2 = \{\,0,\, 1\,\}$. Infinitely many a_n are such that $a_n \notin \mathcal{V}$. In particular, for all odd integers $n \in \mathbb{N}$, $a_n \notin \mathcal{V}$. Does a_n converge to 2? Any open set the contains 2 also contains 1, so given any $\mathcal{U} \in \tau$ with $2 \in \mathcal{U}$, choose $N = 0$. For all $n > N$ we have $a_n \in \mathcal{U}$. So $a_n \to 2$. Also, $a_n \to 3$ and $a_n \to 4$. Moreover, for every integer $k > 1$, $a_n \to k$ is a true statement. ■

Example 2.4 Let X be a set and $\tau = \mathcal{P}(X)$ be the discrete topology. Let $a : \mathbb{N} \to X$ be a sequence. Then given $x \in X$, $a_n \to x$ if and only if there is an $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n = x$. To see this, choose $\mathcal{U} = \{\,x\,\}$. This set is open since it is a subset of X and τ is the discrete topology. Applying the definition of convergence to this set shows that a_n is eventually a constant. ■

Example 2.5 Let $X = \mathbb{R}$ and τ_C be the countable complement topology. If $a : \mathbb{N} \to \mathbb{R}$ converges, then a_n is eventually a constant. We can show that if a_n converges to x, then eventually $a_n = x$ for at least one $n \in \mathbb{N}$. Define $A \subseteq \mathbb{R}$ via:

$$A = \{\,a_n \in \mathbb{R} \mid n \in \mathbb{N}\,\} \tag{15}$$

This is a countable subset, so $\mathbb{R} \setminus A$ is open in the countable complement topology. If $a_n \neq x$ for all $n \in \mathbb{N}$, then $x \in \mathbb{R} \setminus A$. But this is an open set that contains x and never contains any of the a_n, meaning a_n can't possible converge to x. So if $a_n \to x$, then $a_n = x$ for at least one integer $n \in \mathbb{N}$. Now, we can show $a_n = x$ for all sufficiently large n. Define B by:

$$B = \{\,a_n \in \mathbb{R} \mid n \in \mathbb{N} \text{ and } a_n \neq x\,\} \tag{16}$$

This set is also countable, so $\mathbb{R} \setminus B$ is open. Applying the definition of convergence shows that $a_n = x$ for all large $n \in \mathbb{N}$. Contrast this with convergence in the standard topology on $\mathbb{R}$. The sequence $a_n = \frac{1}{n+1}$ converges to zero but is never equal to zero. The countable complement topology does not have such sequences. ■

Uniqueness of limits is given by the Hausdorff property. All of the bizarre examples we've discussed so far involved non-Hausdorff spaces.

Theorem 2.1. *If (X, τ) is a Hausdorff topological space, if $a : \mathbb{N} \to X$ is a convergent sequence, and if $x, y \in X$ are such that $a_n \to x$ and $a_n \to y$, then $x = y$.*

Proof. Suppose not. Since $x \neq y$ and (X, τ) is Hausdorff, there are open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. But $a_n \to x$, so there is an $N_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N_0$ it is true that $a_n \in \mathcal{U}$. But also $a_n \to y$ so there is an $N_1 \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N_1$ implies $a_n \in \mathcal{V}$. Let $N = \max(N_0, N_1)$. Then for all $n \in \mathbb{N}$ with $n > N$ we have $a_n \in \mathcal{U}$ and $a_n \in \mathcal{V}$. But $\mathcal{U} \cap \mathcal{V} = \emptyset$, which is a contradiction. Hence, $x = y$. □

Example 2.6 The converse of this theorem is not true. It is possible for sequences to be unique, but the space to not be Hausdorff. The countable complement topology on $\mathbb{R}$ is an example. ■

Sequences were sufficient to describe open sets in metric spaces. We used the metric, but then proved that a set $\mathcal{U}$ in a metric space (X, d) is open if and only if for every sequence $a : \mathbb{N} \to X$ that converges to some $x \in \mathcal{U}$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $a_n \in \mathcal{U}$. We take this and use it to define *sequentially open* subsets.

Definition 2.2 (**Sequentially Open Subset**) A sequentially open subset in a topological space (X, τ) is a set $\mathcal{U} \subseteq X$ such that for every sequence $a : \mathbb{N} \to X$ that converges to a point $x \in \mathcal{U}$ there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$ it is true that $a_n \in \mathcal{U}$. ■

This is insufficient for topological spaces. We need to use the topology to define openness, not just sequences. The countable complement topology on $\mathbb{R}$ gives us an example. Every subset of $\mathbb{R}$ is sequentially open in the countable complement topology since $a : \mathbb{N} \to \mathbb{R}$ converges if and only if it is eventually constant. However, not every subset of $\mathbb{R}$ is open with the countable complement topology.

Open always implies sequentially open, almost by definition.

Theorem 2.2. *If (X, τ) is a topological space, and if $\mathcal{U} \in \tau$, then $\mathcal{U}$ is sequentially open.*

Proof. For let $a : \mathbb{N} \to X$ be a sequence that converges to $x \in \mathcal{U}$. Then, since $\mathcal{U}$ is open, by the definition of convergence there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $a_n \in \mathcal{U}$. Hence, $\mathcal{U}$ is sequentially open. □

Definition 2.3 (**Sequential Topological Space**) A sequential topological space is a topological space (X, τ) such that for all $\mathcal{U} \subseteq X$, $\mathcal{U}$ is open if and only if $\mathcal{U}$ is sequentially open. ■

Sequential spaces are spaces where sequences are enough. Enough for just about everything. These are spaces where sequences can describe open sets, closed sets, and continuity. It is fortunate that most spaces one encounters are sequential.

3 Continuity

Sequences are not sufficient to describe continuity, since they are not sufficient to describe open sets. In the theory of metric spaces we proved that a function is continuous if and only if the pre-image of an open set is open. This only requires the topology, meaning it is perfect to describe continuity in the general topological setting.

Definition 3.1 (**Continuous Function Between Topological Spaces**) A continuous function from a topological space (X, τ_X) to a topological space (Y, τ_Y) is a function $f : X \to Y$ such that for all $\mathcal{V} \in \tau_Y$ it is true that $f^{-1}[\mathcal{V}] \in \tau_X$. That is, the pre-image of an open set is open. ■

Example 3.1 If Y is a set, $\tau_Y = \{\emptyset, Y\}$ is the indiscrete topology, and if (X, τ_X) is any topological space, then any function $f : X \to Y$ is continuous. We need to check for every open set $\mathcal{V} \in \tau_Y$ that the pre-image $f^{-1}[\mathcal{V}]$ is open. There are only two candidates to check. We have $f^{-1}[\emptyset] = \emptyset$ and $f^{-1}[Y] = X$, both of which are open sets in τ_X. Hence, f is continuous. ■

Example 3.2 If X is a set, $\tau_X = \mathcal{P}(X)$ is the discrete topology, and if (Y, τ_Y) is any topological space, then for any function $f : X \to Y$ it is true that f is continuous. Given any open subset $\mathcal{V} \in \tau_Y$, the pre-image is a subset of X, so $f^{-1}[\mathcal{V}] \in \mathcal{P}(X)$. That is, $f^{-1}[\mathcal{V}] \in \tau_X$, so $f^{-1}[\mathcal{V}]$ is open, and f is continuous. ■

Theorem 3.1. *If (X, τ_X) and (Y, τ_Y) are topological spaces, then $f : X \to Y$ is continuous if and only if for every closed subset $\mathcal{D} \subseteq Y$, the pre-image $f^{-1}[\mathcal{D}]$ is closed in X.*

Proof. Suppose f is continuous, and let $\mathcal{D}$ be closed. Then:

$$f^{-1}[Y \setminus \mathcal{D}] = f^{-1}[Y] \setminus f^{-1}[\mathcal{D}] = X \setminus f^{-1}[\mathcal{D}] \tag{17}$$

But $Y \setminus \mathcal{D}$ is open since $\mathcal{D}$ is closed, so $X \setminus f^{-1}[\mathcal{D}]$ is open. But then $f^{-1}[\mathcal{D}]$ is closed. Now, suppose the pre-image of closed sets are closed. Given $\mathcal{V} \in \tau_Y$, we have:

$$f^{-1}[\mathcal{V}] = f^{-1}[Y \setminus (Y \setminus \mathcal{V})] = f^{-1}[Y] \setminus f^{-1}[Y \setminus \mathcal{V}] = X \setminus f^{-1}[Y \setminus \mathcal{V}] \tag{18}$$

But $\mathcal{V}$ is open, so $Y \setminus \mathcal{V}$ is closed. By assumption $f^{-1}[Y \setminus \mathcal{V}]$ is closed, so $X \setminus f^{-1}[Y \setminus \mathcal{V}]$ is the complement of a closed set, and hence is open. That is, the pre-image of an open set is open, so f is continuous. □

As mentioned, sequences are not enough to describe continuity. We give a new definition to functions that map convergent sequences to convergent sequences.

Definition 3.2 (**Sequentially Continuous Function**) A sequentially continuous function from a topological space $(X,\,\tau_X)$ to a topological space $(Y,\,\tau_Y)$ is a function such that for every convergent sequence $a:\mathbb{N}\to X$ with $x\in X$ such that $a_n\to x$, it is true that $f(a_n)\to f(x)$. ■

There's no requirement that limits be unique in either $(X,\,\tau_X)$ nor $(Y,\,\tau_Y)$. The definition does not need such a notion. Continuity always implies sequential continuity.

Theorem 3.2. *If $(X,\,\tau_X)$ and $(Y,\,\tau_Y)$ are topological spaces, and if $f:X\to Y$ is a continuous function, then f is sequentially continuous.*

Proof. Suppose not. Then there is a sequence $a:\mathbb{N}\to X$ and an $x\in X$ such that $a_n\to x$ but $f(a_n)\not\to f(x)$. But if $f(a_n)\not\to f(x)$, then by the definition of convergence, there is an open set $\mathcal{V}\in\tau_Y$ with $f(x)\in\mathcal{V}$ such that for all $N\in\mathbb{N}$ there is an $n\in\mathbb{N}$ with $n>N$ but $f(a_n)\notin\mathcal{V}$. But $\mathcal{V}$ is open, and f is continuous, so $f^{-1}[\mathcal{V}]$ is open. But since $f(x)\in\mathcal{V}$ it is true that $x\in f^{-1}[\mathcal{V}]$ by the definition of pre-image. But since $f^{-1}[\mathcal{V}]$ is open, $x\in f^{-1}[\mathcal{V}]$, and $a_n\to x$, there is an $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ with $n>N$ it is true that $a_n\in f^{-1}[\mathcal{V}]$. But then $f(a_n)\in\mathcal{V}$ for all $n>N$, which is a contradiction. Hence, f is sequentially continuous. □

Example 3.3 The converse does not reverse, in general. Let $X=Y=\mathbb{R}$, let $\tau_X=\tau_C$, the countable complement topology, and let $\tau_Y=\tau_{\mathbb{R}}$ be the standard topology. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x$. Then f is *not* continuous, but it is sequentially continuous. It is not continuous since $(0,\,1)$ is open in $\tau_{\mathbb{R}}$, but $f^{-1}[(0,\,1)]=(0,\,1)$, and $(0,\,1)$ is not open in τ_C. f is sequentially continuous. If $a:\mathbb{N}\to\mathbb{R}$ converges to $x\in\mathbb{R}$ with respect to τ_C, then there is an $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ with $n>N$ we have $a_n=x$. But then $f(a_n)=x$ for all $n>N$, and therefore $f(a_n)\to f(x)$. ■

Theorem 3.3. *If $(X,\,\tau_X)$ is a sequential topological space, if $(Y,\,\tau_Y)$ is a topological space, and if $f:X\to Y$ is a function, then f is continuous if and only if f is sequentially continuous.*

Proof. Continuity implies sequential continuity in every setting. Let's go the other way. Suppose $f:X\to Y$ is sequentially continuous. Suppose f is *not* continuous. Then there is a $\mathcal{V}\in\tau_Y$ such that $f^{-1}[\mathcal{V}]\notin\tau_X$. That is, there is an open set in Y whose pre-image is not open in X. But $(X,\,\tau_X)$ is sequential, so if $f^{-1}[\mathcal{V}]$ is not open, then it is not sequentially open. But if $f^{-1}[\mathcal{V}]$ is not sequentially open, then there is a sequence $a:\mathbb{N}\to X$ and an $x\in f^{-1}[\mathcal{V}]$ such that $a_n\to x$, but for all $N\in\mathbb{N}$ there is an $n\in\mathbb{N}$ with $n>N$ such that $a_n\notin f^{-1}[\mathcal{V}]$. But f is sequentially continuous, so if $a_n\to x$, then $f(a_n)\to f(x)$. But then $\mathcal{V}$ is an open set containing $f(x)$ and $f(a_n)\to f(x)$, so there is an $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ with $n>N$ it is true that $f(a_n)\in\mathcal{V}$. But then for all $n>N$ it is true that $a_n\in f^{-1}[\mathcal{V}]$, which is a contradiction. Hence, f is continuous. □