Point-Set Topology: Lecture 15

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1 Induced Equivalence Relation

Given a relation R on a set A, it is possible for R to be very dull. It does not need to be reflexive, symmetric, or transitive. We can always transform R into a reflexive relation by adding in aRa for all $a \in A$. We can then make it symmetric by adding bRa for all $a, b \in A$ such that aRb. Lastly, we can make it transitive by enlarging the relation as well. This idea is the *induced equivalence relation* from R.

Theorem 1.1. *If A is a set, and if $\mathcal{R} \subseteq \mathcal{P}(A \times A)$ is a non-empty set such that for all $R \in \mathcal{R}$ it is true that R is an equivalence relation on A, then $\bigcap \mathcal{R}$ is an equivalence relation on A.*

Proof. Given $a \in A$, since for all $R \in \mathcal{R}$ it is true that R is an equivalence relation, we have aRa. Hence, $(a,\, a) \in \bigcap \mathcal{R}$. That is, $a\big(\bigcap \mathcal{R}\big)a$. If $a, b \in A$ are such that $(a,\, b) \in \bigcap \mathcal{R}$, then for all $R \in \mathcal{R}$ we have aRb. But since R is an equivalence relation this implies bRa. So bRa for all $R \in \mathcal{R}$ and therefore $(b,\, a) \in \bigcap \mathcal{R}$. Lastly, if $a, b, c \in A$ are such that $(a,\, b) \in \bigcap \mathcal{R}$ and $(b,\, c) \in \bigcap \mathcal{R}$, then for all $R \in \mathcal{R}$ we have aRb and bRc. But R is an equivalence relation, so then aRc. But then $(a,\, c) \in \bigcap \mathcal{R}$, so $\bigcap \mathcal{R}$ is transitive. Hence, $\bigcap \mathcal{R}$ is an equivalence relation. □

Theorem 1.2. *If A is a set, if R is a relation on A, and if $\mathcal{R}$ is the set of all equivalence relations R' on A such that $R \subseteq R'$, then $\bigcap \mathcal{R}$ is an equivalence relation on A such that $R \subseteq \bigcap \mathcal{R}$.*

Proof. First, $\mathcal{R}$ is non-empty since $A \times A$ is an equivalence relation on A. It is the relation that says a is related to b for all $a, b \in R$. That is, the relation that says everything is related to everything else. But R is a relation on A, so by definition $R \subseteq A \times A$. Hence $\mathcal{R}$ is a non-empty set of equivalence relations on A, so $\bigcap \mathcal{R}$ is an equivalence relation on A by the previous theorem. But for all $R' \in \mathcal{R}$ it is true that $R \subseteq R'$, so $R \subseteq \bigcap \mathcal{R}$. □

Definition 1.1 (**Induced Equivalence Relation**) The induced equivalence relation on a set A by a relation R is the equivalence relation $\bigcap \mathcal{R}$ where $\mathcal{R}$ is the set of all equivalence relations R' on A such that $R \subseteq R'$. ■

Example 1.1 Let A be a set and $R = \emptyset$, the empty relation. This relation says nothing is related, not even $a \in A$ is related to itself. The induced equivalence relation is the diagonal $\Delta_A = \{\, (a,\, a) \mid a \in A \,\}$. The only thing we need to add to make R an equivalence relation is reflexivity. ■

Theorem 1.3. *If A is a set, and if R is an equivalence relation on A, then the induced equivalence relation R' is equal to R.*

Proof. Let $\mathcal{R}$ be the set of all equivalence relations R'' on A such that $R \subseteq R''$. But R is an equivalence relation on A, and $R \subseteq R$, so $R \in \mathcal{R}$. Hence $\bigcap \mathcal{R} \subseteq R$. But also $R \subseteq \bigcap \mathcal{R}$. So $R = \bigcap \mathcal{R}$. But $R' = \bigcap \mathcal{R}$ since R' is the induced equivalence relation, so $R = R'$. □

Definition 1.2 (**Induced Equivalence Relation by a Subset**) The induced equivalence relation of a subset $A \subseteq X$ of a set X is the induced equivalence relation R_A induced by the relation R on X defined by:

$$R = \{\, (a,\, b) \in X \times X \mid a \in A \text{ and } b \in A \,\} \tag{1}$$

That is, the equivalence relation induced by saying that everything in A is related to everything else in A. ■

2 Quotient Topology

Given a set X and an equivalence relation R on X, we may form the quotient set X/R which is the set of all equivalence classes of X under R. Intuitively, we are taking points in X and *gluing* them together in the quotient set. If X had a topology, it seems like it should be possible to give a topology to the quotient since gluing things together certainly seems like a topological operation. This is indeed possible, and quotient spaces are very common in topology since they provide a plethora of spaces one can ponder and construct. We define the quotient topology via the quotient map. In set theory, there is a canonical quotient function $q : X \to X/R$ given by $q(x) = [x]$ for all $x \in X$, where $[x] \in X/R$ is the equivalence class of x. This is something like projecting points x in X to the point in X/R where x was glued to, the equivalence class $[x]$. This gluing operation should be continuous. We define the quotient topology on X/R via the *final topology* on X/R which makes q continuous.

Definition 2.1 (**Quotient Topology**) The quotient topology on the quotient set X/R of a set X under an equivalence relation R with respect to a topological space $(X,\, \tau)$ is the set $\tau_{X/R}$ defined as the final topology with respect to the quotient map $q : X \to X/R$ defined by $q(x) = [x]$, and with respect to the topology τ on X. ■

Theorem 2.1. *If $(X,\, \tau)$ is a topological space, if R is an equivalence relation on X, and if $\tau_{X/R}$ is the quotient topology on X/R, then $(X/R,\, \tau_{X/R})$ is a topological space.*

Proof. The quotient topology is the final topology with respect to (X, τ) and the quotient map $q : X \to X/R$ defined by $q(x) = [x]$. But the final topology for any function $f : X \to X/R$ with respect to (X, τ) is a topology on X/R, so in particular $\tau_{X/R}$ is a topology. $\square$

Since $\tau_{X/R}$ is the final topology with respect to the quotient mapping q, a subset $\mathcal{U} \subseteq X/R$ is open *if and only if* $q^{-1}[\mathcal{U}]$ is open. Please note continuity alone is not sufficient enough to say that $q^{-1}[\mathcal{U}]$ being open implies $\mathcal{U}$ is open. The implication goes one way. If $\mathcal{U}$ is open, and if q is continuous, then $q^{-1}[\mathcal{U}]$ is open. For a general continuous function $f : X \to Y$ with topologies τ_X and τ_Y, given $\mathcal{V} \subseteq Y$ and $f^{-1}[\mathcal{V}] \in \tau_X$, it is not necessarily true that we can conclude that $\mathcal{V} \in \tau_Y$. Let $X = Y = \mathbb{R}$, and $\tau_X = \tau_Y = \tau_{\mathbb{R}}$, the standard topology on $\mathbb{R}$. Let $f(x) = 1$. Since it is a constant function, it is continuous. But $f^{-1}[\{\,1\,\}] = \mathbb{R}$, which is open, however $\{\,1\,\}$ is not open in $\mathbb{R}$. The quotient map, with the quotient topology, is very special in this regard. $\mathcal{U} \subseteq X/R$ is open *if and only if* $q^{-1}[\mathcal{U}]$ is open in X. This fact is used constantly in the proofs of various claims about quotient spaces.

Definition 2.2 (**Saturated Subset**) A saturated set with respect to a function $f : X \to Y$ between sets X and Y is a set $A \subseteq X$ such that $f^{-1}\big[f[A]\big] = A$. $\blacksquare$

Not every subset is saturated (unless the function is injective). We can always conclude that $A \subseteq f^{-1}\big[f[A]\big]$, however.

Theorem 2.2. *If X and Y are sets, if $f : X \to Y$, and if $A \subseteq X$, then $A \subseteq f^{-1}\big[f[A]\big]$.*

Proof. Given $x \in A$ it is true that $f(x) \in f[A]$ by the definition of image. But then x is an element of X such that $f(x) \in f[A]$, and hence $x \in f^{-1}\big[f[A]\big]$ by the definition of pre-image. So $A \subseteq f^{-1}\big[f[A]\big]$. $\square$

Theorem 2.3. *If X and Y are sets, if $f : X \to Y$ is an injective function, and if $A \subseteq X$, then $A = f^{-1}\big[f[A]\big]$.*

Proof. We have proven that $A \subseteq f^{-1}\big[f[A]\big]$. Let's go the other way. Let $x \in f^{-1}\big[f[A]\big]$. Then there is a $y \in f[A]$ such that $f(x) = y$. But $y \in f[A]$, so there is an element $x_0 \in A$ such that $f(x_0) = y$. But f is injective, so $x = x_0$. Therefore, $x \in A$. That is, $f^{-1}\big[f[A]\big] \subseteq A$, so $A = f^{-1}\big[f[A]\big]$. $\square$

Lacking injectivity, we can make no such conclusion.

Example 2.1 Let $X = Y = \mathbb{R}$, and let $A = \mathbb{R}_{\geq 0}$. Define $f(x) = x^2$. Then $f^{-1}\big[f[A]\big] = \mathbb{R}$, but $A \neq \mathbb{R}$. $\blacksquare$

Theorem 2.4. *If X and Y are sets, if $f : X \to Y$ is a function, and if $B \subseteq Y$, then $A = f^{-1}[B]$ is a saturated subset of X.*

Proof. We have proven that $A \subseteq f^{-1}\big[f[A]\big]$. Going the other way, let $x \in f^{-1}\big[f[A]\big]$. Then $f(x) \in f[A]$ by the definition of pre-image. But $f[A] =$

$f\left[f^{-1}[B]\right]$ by the definition of A. So if $f(x) \in f[A]$, then $f(x) \in f\left[f^{-1}[B]\right]$. But then $f(x) \in B$. But if $f(x) \in B$, then $x \in f^{-1}[B]$. Therefore $x \in A$, so $f^{-1}\left[f[A]\right] \subseteq A$, and hence $f^{-1}\left[f[A]\right] = A$. That is, A is saturated. □

Theorem 2.5. *If (X, τ) is a topological space, if R is an equivalence relation on X, and if $\tau_{X/R}$ is the quotient topology, then the quotient map $q : X \to X/R$ is a continuous surjective function such that for all saturated $\mathcal{U} \in \tau$, it is true that $q[\mathcal{U}] \in \tau_{X/R}$.*

Proof. Since q is the final topology with respect to (X, τ) and q, q is continuous. q is also surjective, since given $[x] \in X/R$ we have $q(x) = [x]$. Lastly, if $\mathcal{U} \in \tau$ is saturated, then $q^{-1}\left[q[\mathcal{U}]\right] = \mathcal{U}$. But then $q[\mathcal{U}]$ is a set in X/R such that the pre-image is an open subset of X, and since $\tau_{X/R}$ is the quotient topology, it must be true that $q[\mathcal{U}]$ is open. Hence the image of a saturated open set is open. □

This theorem motivates the more general idea of a quotient map between different topological spaces.

Definition 2.3 (**Quotient Map**) A quotient map from a topological space (X, τ_X) to a topological space (Y, τ_Y) is a continuous surjective function $f : X \to Y$ such that for every saturated set $\mathcal{U} \in \tau_X$ it is true that $f[\mathcal{U}] \in \tau_Y$. ■

Theorem 2.6. *If (X, τ_X) and (Y, τ_Y) are topological spaces, if $f : X \to Y$ is a quotient map, then there is an equivalence relation R on X such that (Y, τ_Y) is homeomorphic to $(X/R, \tau_{X/R})$ where $\tau_{X/R}$ is the quotient topology.*

Proof. Let R be the relation on X defined by aRb if and only if $f(a) = f(b)$. R is an equivalence relation. aRa since $f(a) = f(a)$. R is symmetric since aRb implies $f(a) = f(b)$, and hence $f(b) = f(a)$, so bRa. Lastly, R is transitive. If aRb and bRc, then $f(a) = f(b)$ and $f(b) = f(c)$. By the transitivity of equality, $f(a) = f(c)$ and hence aRc. Define $g : X/R \to Y$ via $g([a]) = f(a)$. This is well-defined. If $[a] = [b]$, then aRb, and hence $f(a) = f(b)$. Thus $g([a]) = f(a) = f(b) = g([b])$. We now must prove that g is a homeomorphism. First, it is bijective. It is injective since if $g([a]) = g([b])$, then $f(a) = f(b)$, and hence aRb, so $[a] = [b]$. It is surjective since given $y \in Y$, since f is a quotient map it is surjective, so there is an $x \in X$ such that $f(x) = y$. But then $g([x]) = f(x) = y$, so g is surjective. Therefore, g is bijective. Next, to prove g is a continuous open map. Given $\mathcal{V} \in \tau_Y$, $g^{-1}[\mathcal{V}]$ is the set of all $[x] \in X/R$ such that $g([x]) \in \mathcal{V}$. But $g([x]) \in \mathcal{V}$ if and only if $f(x) \in \mathcal{V}$, and $f(x) \in \mathcal{V}$ if and only if $x \in f^{-1}[\mathcal{V}]$. But f is a quotient map, so it is continuous, and hence $f^{-1}[\mathcal{V}]$ is open. But $f^{-1}[\mathcal{V}] = q^{-1}\left[g^{-1}[\mathcal{V}]\right]$ where $q : X \to X/R$ is the quotient map $q(x) = [x]$. But $\tau_{X/R}$ is the quotient topology, so if $q^{-1}\left[g^{-1}[\mathcal{V}]\right]$ is open, then $g^{-1}[\mathcal{V}]$ is open, so g is continuous. Lastly, g is an open mapping. Given $\mathcal{U} \in \tau_{X/R}$, $q^{-1}[\mathcal{U}]$ is a saturated open subset of X. But f is a quotient map, so then $f\left[q^{-1}[\mathcal{U}]\right]$ is open. But $f\left[q^{-1}[\mathcal{U}]\right] = g[\mathcal{U}]$, so $g[\mathcal{U}]$ is open. Therefore g is a homeomorphism. □

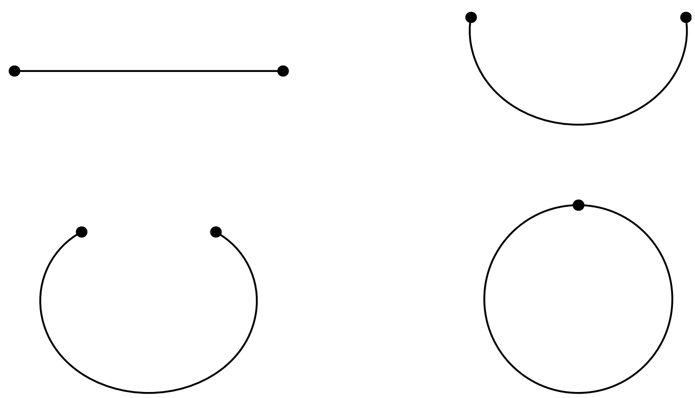

Figure 1: Quotient of an Interval to a Circle

Quotient spaces (Y, τ_Y) are just topological spaces that can be thought of as quotients by some equivalence relation of another topological (X, τ_X). This mimics how topological embeddings give us topological spaces that can be thought of as subspaces of some other topological space.

3 Quotient of a Subspace

The most common way to create a quotient of a topological space (X, τ) is to take a subset $A \subseteq X$, give the equivalence relation R_A on X that is induced by A, and consider X/R_A with the quotient topology τ_{X/R_A}. The notation for this is quite unfortunate, we write X/A and $\tau_{X/A}$. The reason this is unfortunate is because we now have competing notation with algebraists. Given $\mathbb{R}$ with the standard topology, and $\mathbb{Z} \subseteq \mathbb{R}$, algebraists will tell you that $\mathbb{R}/\mathbb{Z}$ is a circle. Topologists will tell you this is actually infinitely many circles. The reason for the competing notions is that, to algebraists, $\mathbb{R}$ is a *group*, $\mathbb{Z} \subseteq \mathbb{R}$ is a *subgroup*, and $\mathbb{R}/\mathbb{Z}$ is a *quotient group*, which is indeed the same thing as a circle, as far as groups are concerned. For topologists, $\mathbb{R}$ is a *topological space*, $\mathbb{Z} \subseteq \mathbb{R}$ is a *topological subspace*, and $\mathbb{R}/\mathbb{Z}$ is a *quotient space*, which we will see later looks like infinitely many circles all touching at one point.

Example 3.1 Let $X = [0,\, 1]$, τ_X the subspace topology from $\mathbb{R}$, and $A = \{\, 0,\, 1\,\}$. The quotient space X/A is formed by taking the endpoints of X and gluing them together. The result is a circle. A visual for is given in Fig. 1. Note, the quotient space is not exactly a circle, it is just homeomorphic to it. The points in $\mathbb{S}^1$ are points in the plane $\mathbb{R}^2$. Points in X/A are *equivalence classes of* X, which means points in X/A are subsets $[x] \subseteq X$ for $x \in X$. The homeomorphism goes as follows. Define:

$$f([t]) = \big(\cos(2\pi t),\, \sin(2\pi t)\big) \tag{2}$$

$[0] = [1]$ since the equivalence relation glues 0 to 1, meaning this function is

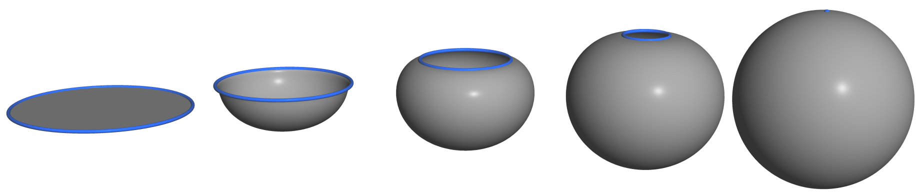

Figure 2: Quotient of a Disk to a Sphere

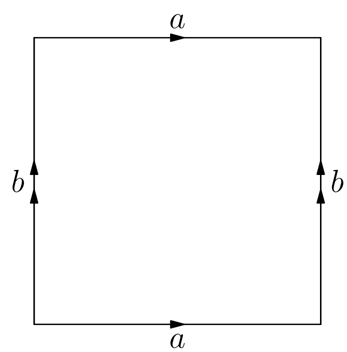

Figure 3: Square Representation of a Torus

indeed bijective, and it is also continuous with a continuous inverse. ■

Example 3.2 Let $X \subseteq \mathbb{R}^2$ be the closed unit disk:

$$X = \{\, \mathbf{x} \in \mathbb{R}^2 \mid ||\mathbf{x}||_2 \leq 1 \,\} \tag{3}$$

This includes the *boundary*, the points that are precisely 1 unit away from the origin. Let $A \subseteq X$ be the unit circle, $A = \mathbb{S}^1$:

$$A = \{\, \mathbf{x} \in \mathbb{R}^2 \mid ||\mathbf{x}||_2 = 1 \,\} \tag{4}$$

Equip both X and A with the subspace topologies from the Euclidean plane. The quotient space X/A is a sphere. We are taking the points on the boundary and gluing them to a single point. This is shown in Fig. 2. Again, X/A is not exactly the sphere in $\mathbb{R}^3$, it is just homeomorphic to it. The sphere contains points in $\mathbb{R}^3$ whereas X/A contains subsets of X, the equivalence classes of X under the relation that glues A to a single point. Topologically, however, there is little point in differentiating between X/A and the sphere $\mathbb{S}^2$ since they are homeomorphic. ■

Consider the square $[0,\,1] \times [0,\,1]$. Identity $(x,\,0)$ with $(x,\,1)$ for all $x \in [0,\,1]$, and also $(0,\,y)$ with $(1,\,y)$ for all $y \in [0,\,1]$. This identification is shown in Fig. 3. The quotient of the square under this identication is a torus, which is a hollow donut.

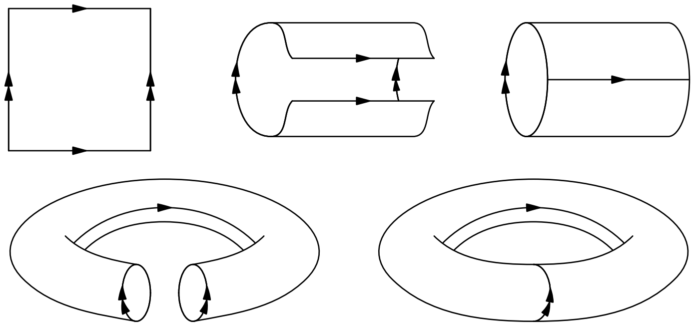

Figure 4: Quotient of a Square to a Torus

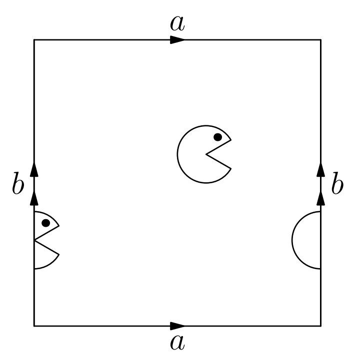

Figure 5: Pac-Man's World

By gluing the top edge to the bottom edge we obtain a cylinder. The left and right edges become circles in the process, and we now have to glue these circles together with matching orientations. By doing this we obtain a torus. This is shown in Fig. 4.

The torus is the world the Pac-Man lives on (See Fig. 5). Does Pac-Man see his own back? We can tile the plane with squares, so let's take a copy of Fig. 3 and use it to cover the page, ensuring that the orientation of the arrows match when we glue adjacent squares together. The result is Fig. 6. In this figure the two Pac-Men are given different colors so we can differentiate them. This idea of tiling the plane with the square representation of the torus will be very important later. It shows that the plane is the *universal cover* of the torus, a concept that is fundamental to algebraic topology and the theory of manifolds.

Now, let's do a different identification on the square. Let's identify $(0, y)$ with $(1, 1-y)$. That is, we are taking the square and gluing the left and right sides

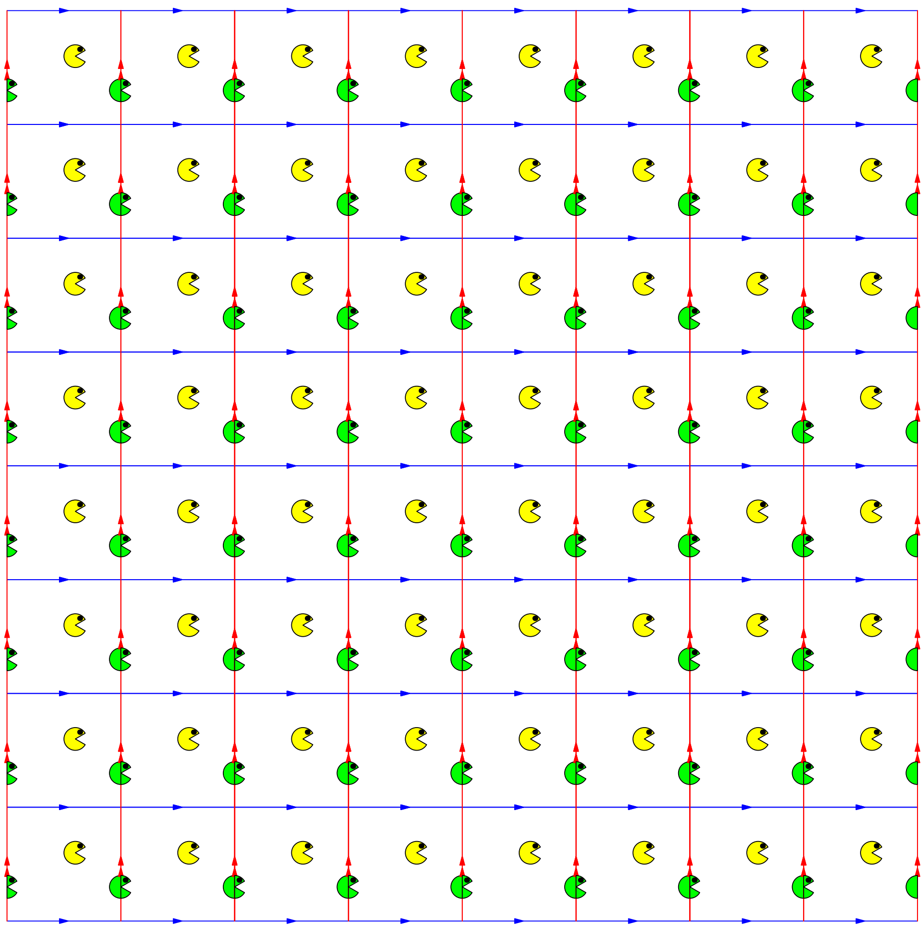

Figure 6: Tiling the Plane with Pac-Man's World

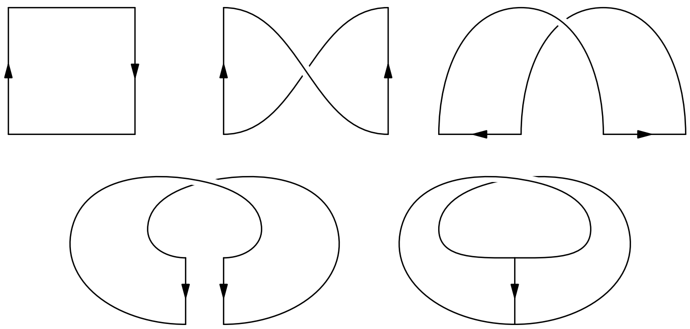

Figure 7: Square Representation of the Möbius Strip

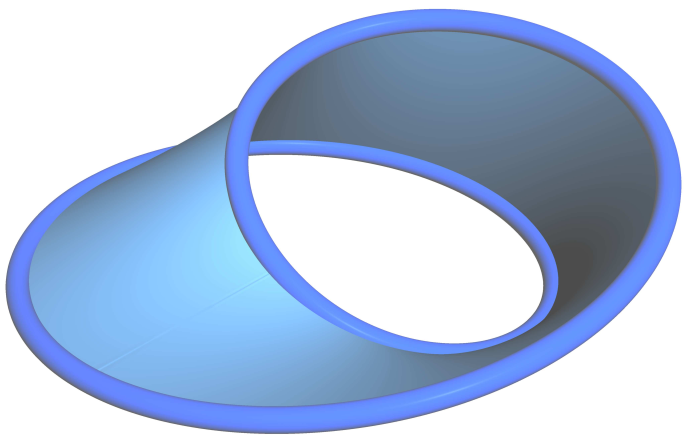

Figure 8: A Möbius Strip

together, but with a twist. The result is the Möbius strip, this construction is shown in Fig. 7. A 3D drawing is shown in Fig. 8.

And now let's go nuts. Let identify $(x, 0)$ with $(1 - x, 1)$, like in the Möbius band, but also $(0, y)$ with $(1, y)$, like in the torus. The result is something that's like a torus, but also like a Möbius band. The square representation is given in Fig. 9. This object is called the *Klein bottle*.

Let's imagine Pac-Man lived on a Klein bottle, instead of a torus. This is shown in Fig. 10. We know that Pac-Man will see his back due to the torus-like identification made with the left and right edges, but will Pac-Man also see his face? If we play the same game as before, taking copies of Fig. 10 and attach them to tile the plane in a consistent manner, we can see that Pac-Man does indeed see this face. This is shown in Fig. 11.

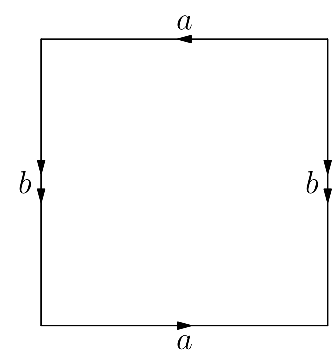

Figure 9: Square Representation of a Klein Bottle

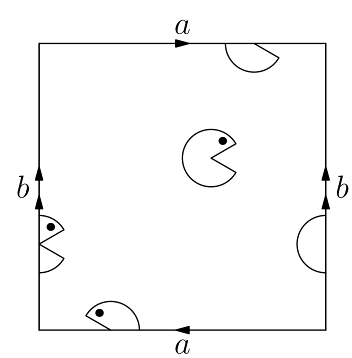

Figure 10: Pac-Man in a Klein Bottle

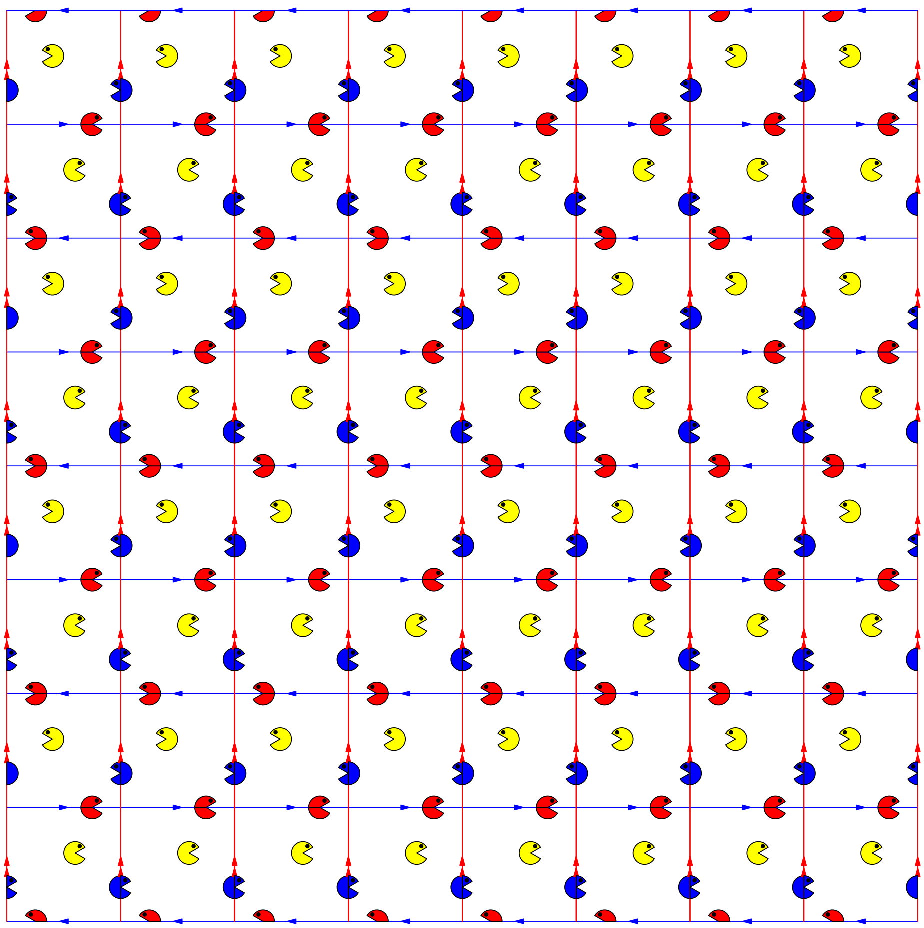

Figure 11: Tiling the Plane with the Klein Bottle

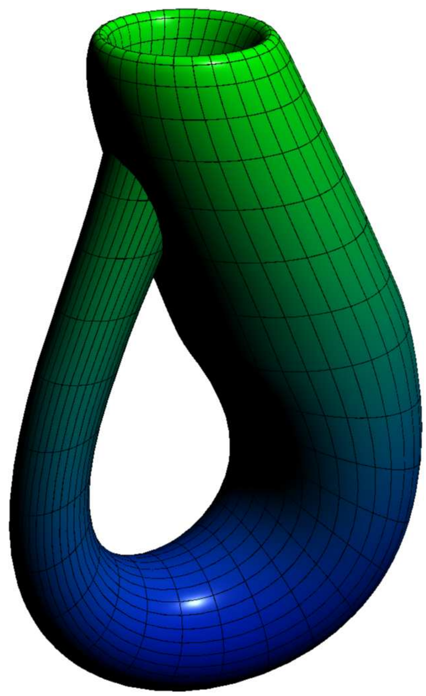

Figure 12: The Klein Bottle in $\mathbb{R}^3$

This shows the plane is also the universal cover of the Klein bottle as well. This idea actually allows us to *immerse* (which is a weaker notion than embed) the Klein bottle into $\mathbb{R}^3$. It is impossible to embed the Klein bottle into $\mathbb{R}^3$ since you will need the object to pass through itself, which is not an embedding. This is given in Fig. 12.

Let's end with the *real projective plane*. This is denoted $\mathbb{RP}^2$. Take the square and identify $(0, y)$ with $(1, 1-y)$, and $(x, 0)$ with $(1-x, 1)$. That is, do the Möbius twist for both top and bottom, and left and right. This is shown in Fig. 13. Can we tile the plane with this object? Let's try. Fig. 14 seems to do it, but there's a cheat. We are not just using copies of Fig. 13, rather we are using copies of Fig. 13 and its mirror. This is why the horizontal arrows converge to the center, and the vertical arrows diverge. Using only copies of Fig. 13 (no mirrors), it is not possible to tile the plane in a way that the arrows match. The reason being that the real projective plane does **not** have the plane as its universal cover. The universal cover of $\mathbb{RP}^2$ is $\mathbb{S}^2$, the sphere. This can be described using a quotient. On $\mathbb{S}^2$, define $\mathbf{x}R\mathbf{y}$ if and only if $\mathbf{y} = -\mathbf{x}$ or $\mathbf{y} = \mathbf{x}$. This is the *antepodal* identification. We are gluing opposite ends of the sphere together. For example, the north pole is glued to the south pole. The result of the quotient $\mathbb{S}^2/R$ is the real projective plane.

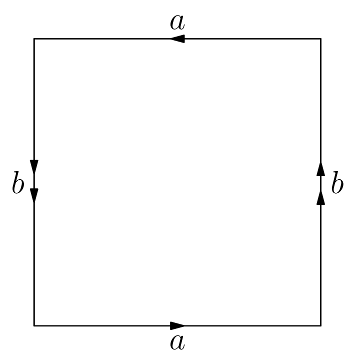

Figure 13: Square Representation of $\mathbb{RP}^2$

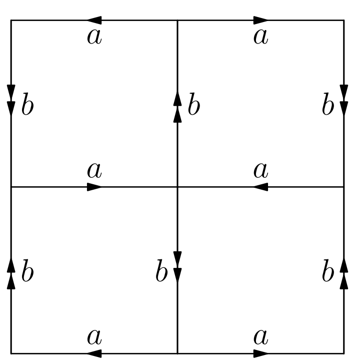

Figure 14: Fake Tiling of the Plane with the $\mathbb{RP}^2$ Square

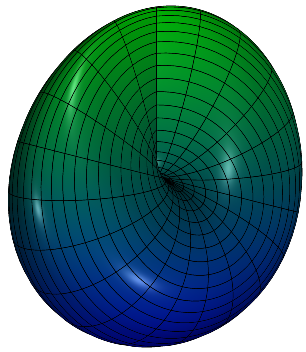

Figure 15: The Cross-Cap $\mathbb{RP}^2$

Like the Klein bottle, it is not possible to embed $\mathbb{RP}^2$ into $\mathbb{R}^3$. We can draw $\mathbb{RP}^2$ if we allow the object to intersect itself. There are many ways to do this, but two nice drawings are given by the so-called *cross-cap* and the *Bryant-Kusner parameterization*. These are given in Figs. 15 and 16, respectively.

4 Properties of Quotients

Most topological properties are **not** preserved by quotients. For example, just because $(X,\,\tau)$ is Hausdorff, doesn't mean all of it's quotient spaces are. The easiest example to describe is the *bug-eyed line*, also known as the line with two origins. Take $X\subseteq\mathbb{R}^2$ to be:

$$X=\{\,(x,\,y)\in\mathbb{R}^2\mid y=-1\text{ or }y=1\,\}\tag{5}$$

Equip this with the subspace topology from $\mathbb{R}^2$. Define R to be the equivalence relation induced by identifying $(x,\,-1)$ with $(x,\,1)$ for all $x\neq 0$. Do not identify $(0,\,-1)$ and $(0,\,1)$ together, keep them separate. Consider the quotient space X/R. First note that since X is a subspace of $\mathbb{R}^2$, which is Hausdorff, $(X,\,\tau_X)$ is Hausdorff as well (τ_X being the subspace topology). The visual is given in Fig. 17. The top part is X, the center is X/R, and the bottom is how we intuitively try to think of X/R, though realizing the middle picture is slightly more accurate. What do open subsets around the two origins look like? A subset of the bug-eyed line is open if and only if the pre-image of the set is an open subset of the subspace $X\subseteq\mathbb{R}^2$ via the quotient map $q:X\rightarrow X/R$. Using this we see that open sets can *look like* open intervals. In particular, if we look at the *top* origin, we can put an open interval around it that does not include the bottom origin. Similarly we can put an open interval around the *bottom* interval that does not include the top. This is done in Fig. 18. The open sets $\mathcal{U}$ and $\mathcal{V}$ in this figure help show that the bug-eyed line is a Fréchet topological space, but it is not Hausdorff. Any open set $\mathcal{U}$ that contains the top origin

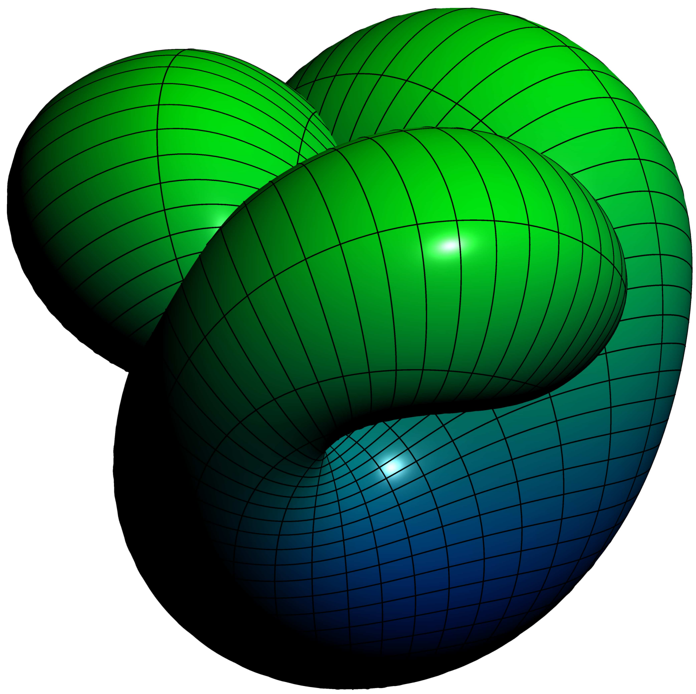

Figure 16: Bryan-Kusner Parameterization of $\mathbb{RP}^2$

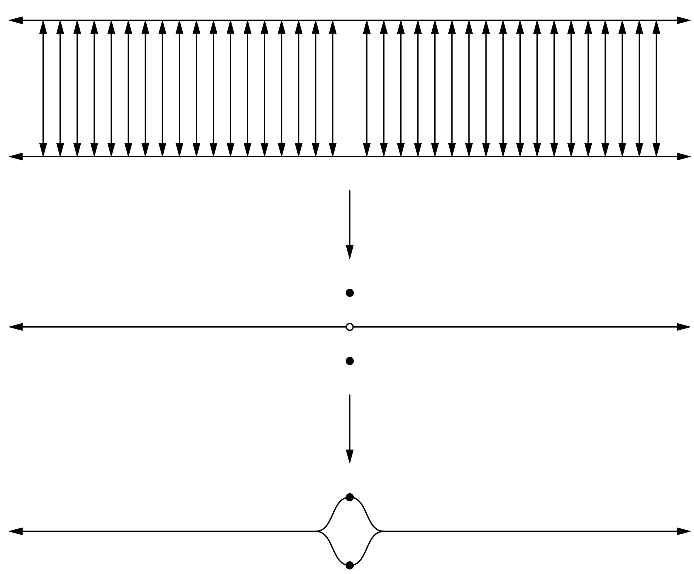

Figure 17: The Bug-Eyed Line

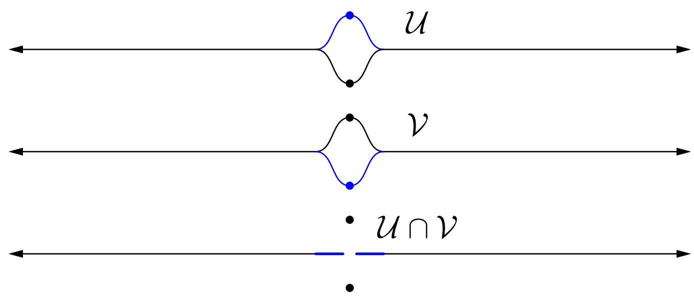

Figure 18: Open Subsets in the Bug-Eyed Line

must overlap with any open set $\mathcal{V}$ that contains the bottom origin (again, see Fig. 18). Quotients of Hausdorff spaces do not need to be Hausdorff.

Quotients do no need to preserve first or second-countable, either. Give $\mathbb{R}$ the standard topology, and consider the quotient space $\mathbb{R}/\mathbb{Z}$. This is **not** the same as the quotient group in abstract algebra, where $\mathbb{R}/\mathbb{Z}$ is just a circle, this is a topological quotient. We are taking all of the integers and gluing them to 0. The result, intuitively, is infinitely many circles that are all touching at 0. $\mathbb{R}$ is second-countable, and hence first-countable, but $\mathbb{R}/\mathbb{Z}$ is neither. To show this, I'll demonstrate that $[0]$, the equivalence class of 0, has no countable neighborhood basis. For let $\tau_{\mathbb{R}}$ be the standard topology and $\tau_{\mathbb{R}/\mathbb{Z}}$ the quotient topology on $\mathbb{R}/\mathbb{Z}$, and let $\mathcal{B} \subseteq \tau_{\mathbb{R}/\mathbb{Z}}$ be any countable collection such that $[0] \in \mathcal{U}$ for all $\mathcal{U} \in \mathcal{B}$. Since $\mathcal{B}$ is countable, there is a surjection $\mathcal{U} : \mathbb{N} \to \mathcal{B}$ so that we may list the elements of $\mathcal{B}$ as:

$$\mathcal{B} = \{\mathcal{U}_0, \ldots, \mathcal{U}_n, \ldots\} \tag{6}$$

Let $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map $q(x) = [x]$. Since each $\mathcal{U}_n$ is open, $q^{-1}[\mathcal{U}_n]$ is open. But $[0] \in \mathcal{U}_0$, so $0 \in q^{-1}[\mathcal{U}_0]$. But since $q^{-1}[\mathcal{U}_0]$ is open, there is an $0 < \varepsilon_0 < 1/2$ such that $|y| < \varepsilon_0$ implies $y \in q^{-1}[\mathcal{U}_0]$. Let $\mathcal{V}_1$ be the $\varepsilon_0/2$ ball centered at 0. Now, since $[0] \in \mathcal{U}_1$ and $[0] = [1]$, we have that $[1] \in \mathcal{U}_1$. But then $1 \in q^{-1}[\mathcal{U}_1]$. Since $1 \in q^{-1}[\mathcal{U}_1]$ and $q^{-1}[\mathcal{U}_1]$ is open, there is a $0 < \varepsilon_1 < 1/2$ such that $|1 - y| < \varepsilon_1$ implies $y \in q^{-1}[\mathcal{U}_1]$. Let $\mathcal{V}_1$ be the $\varepsilon_1/2$ ball centered at 1. Inductively, since $[0] = [n]$ and $[0] \in \mathcal{U}_n$, we have $[n] \in \mathcal{U}_n$ for all $n \in \mathbb{N}$. But then $n \in q^{-1}[\mathcal{U}_n]$. But $q^{-1}[\mathcal{U}_n]$ is open, so there is a $0 < \varepsilon_n < 1/2$ such that $|y - n| < \varepsilon_n$ implies $y \in q^{-1}[\mathcal{U}_n]$. Let $\mathcal{V}_n$ be the $\varepsilon_n/2$ ball centered about n. Let $\mathcal{W} = \bigcup_{n=0}^{\infty} \mathcal{V}_n \cup (-\infty, -1/2)$. Since $\mathcal{W}$ is the union of open subsets of $\mathbb{R}$, it is open. This set is also saturated, so $q[\mathcal{W}] \subseteq \mathbb{R}/\mathbb{Z}$ is open. By construction there is no $\mathcal{U}_n \in \mathcal{B}$ such that $\mathcal{U}_n \subseteq q[\mathcal{W}]$, even though $[0] \in q[\mathcal{W}]$. Hence $\mathcal{B}$ is not a neighborhood basis for $[0]$ and $(\mathbb{R}/\mathbb{Z}, \tau_{\mathbb{R}/\mathbb{Z}})$ is not first-countable, and hence not second-countable either.

The sequential property *is* preserved by quotients.

Theorem 4.1. *If (X, τ) is a sequential topological space, if R is an equivalence* *relation on X, and if $\tau_{X/R}$ is the quotient topology on X/R, then $(X/R,\, \tau_{X/R})$ is sequential.*

Proof. Suppose not. Then there is a sequentially open subset $\mathcal{U} \subseteq X/R$ that is not open. But q is a quotient map, so if $\mathcal{U}$ is not open, then $q^{-1}[\mathcal{U}]$ is not open. But $(X,\, \tau)$ is sequential, so if $q^{-1}[\mathcal{U}]$ is not open, then it is not sequentially open. But then there is a point $x \in q^{-1}[\mathcal{U}]$ and a sequence $a : \mathbb{N} \to X$ such that $a_n \to x$ but for all $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ with $n > N$ and $a_n \notin q^{-1}[\mathcal{U}]$. But since $a_n \to x$ and q is continuous, we have $q(a_n) \to q(x)$. But $q(x) \in \mathcal{U}$ and $\mathcal{U}$ is sequentially open, so there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$ we have $q(a_n) \in \mathcal{U}$. But the for all $n > N$ we have $a_n \in q^{-1}[\mathcal{U}]$, a contradiction. So $\mathcal{U}$ is open, and $(X/R,\, \tau_{X/R})$ is a sequential topological space. $\square$

Two more vital properties are preserved by quotients, but we haven't gotten to them yet. The quotient of a *connected* space is still connected, and the quotient of a *compact* is still compact. We'll discuss both ideas for topological spaces in due time.