Point-Set Topology: Lecture 17

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1 Order Topology and Orderable Spaces

A total order on a set X is a relation $\leq$ that is reflexive ($a \leq a$), anti-symmetric ($a \leq b$ and $b \leq a$ implies $a = b$), transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$), and total (either $a \leq b$ or $b \leq a$ for all a and b). This induces another relation $<$ on X defined by $a < b$ if and only if $a \leq b$ and $a \neq b$. The primary example is *less than or equal to* on $\mathbb{R}$, and the induced relation is *less than*. Given a total order on a set X it is possible to use this to induce a topology $\tau_{<}$ on X that has some very nice properties. Many topological properties of the real line stem from the fact that the standard topology $\tau_{\mathbb{R}}$ and the order topology $\tau_{<}$ are the same (where $<$ is the usual *less than* relation). First, some notation. Given a totally ordered set $(X, <)$, and $a, b \in X$, we write:

$$(a,\, b) = \{\, c \in X \mid a < c \text{ and } c < b \,\} \tag{1}$$
$$[a,\, b) = \{\, c \in X \mid a \leq c \text{ and } c < b \,\} \tag{2}$$
$$(a,\, b] = \{\, c \in X \mid a < c \text{ and } c \leq b \,\} \tag{3}$$
$$(-\infty,\, a) = \{\, c \in X \mid c < a \,\} \tag{4}$$
$$(a,\, \infty) = \{\, c \in X \mid a < c \,\} \tag{5}$$
$$(-\infty,\, a] = \{\, c \in X \mid c \leq a \,\} \tag{6}$$
$$[a,\, \infty) = \{\, c \in X \mid a \leq c \,\} \tag{7}$$

Note, we're not saying ∞ is a thing, or an element of X, this is just notation. Just like how $(0,\, \infty)$ is the set of all positive numbers in $\mathbb{R}$, even though ∞ is not a number. We use this to define the order topology.

Definition 1.1 (**Order Topology**) The order topology on a totally ordered set $(X, <)$ is the topology $\tau_{<}$ generated by the set $\mathcal{B}$ defined by:

$$\mathcal{B} = \{\, (a,\, b) \mid a, b \in X \,\} \cup \{\, (a,\, \infty) \mid a \in X \,\} \cup \{\, (-\infty,\, a) \mid a \in X \,\} \tag{8}$$

That is, the set of all open intervals, open right-rays, and open left-rays. ■

Example 1.1 The real line $\mathbb{R}$ with the standard Euclidean topology is also the order topology induced by the *less than* relation. The Euclidean metric on $\mathbb{R}$ yields a basis consisting of open intervals, which is precisely the order topologies basis. ■

Definition 1.2 (*Linearly Orderable Topological Space*) A linearly orderable topological space is a topological space $(X,\, \tau)$ such that there exists a total order $\leq$ on X such that $\tau = \tau_{<}$ where $\tau_{<}$ is the order topology. ■

Like metrizable spaces, linearly orderable spaces are very nice, topologically.

Theorem 1.1. *If $(X,\, \tau)$ is a linearly orderable topological space, then it is Hausdorff.*

Proof. Let $a, b \in X$, $a \neq b$, and let $<$ be the order that induces τ. Since $<$ comes from a total order, either $a < b$ or $b < a$. Suppose $a < b$ (the proof is symmetric). If there are no elements $c \in X$ such that $c \in (a,\, b)$, then let $\mathcal{U} = (-\infty,\, b)$ and $\mathcal{V} = (a,\, \infty)$. Then $a \in \mathcal{U}$ and $b \in \mathcal{V}$ since $a < b$. But also $\mathcal{U}$ and $\mathcal{V}$ are open by the definition of the order topology. Moreover, $\mathcal{U} \cap \mathcal{V} = \emptyset$ since there are no elements c such that $a < c$ and $c < b$. If there is an element $c \in (a,\, b)$, let $\mathcal{U} = (-\infty,\, c)$ and $\mathcal{V} = (c,\, \infty)$. Then $a \in \mathcal{U}$, $b \in \mathcal{V}$, and $\mathcal{U}$ and $\mathcal{V}$ are open. Also $\mathcal{U} \cap \mathcal{V} = \emptyset$ since you can't have $x < c$ and $c < x$ simultaneously. Hence, $(X,\, \tau)$ is Hausdorff. □

Example 1.2 (**Subspace Order**) Given a totally ordered space $(X,\, \leq)$ with the order topology $\tau_{<}$, if we have $A \subseteq X$ there are two topologies we can place on A. First, the subspace topology from $\tau_{<}$. Second, $\leq$ restricts to a total order on A, label this $\leq_A$. We get a topology $\tau_{<_A}$ via this subspace order. It does **not** need to be the case that the subspace topology and the suborder topologies are the same. Let $A \subseteq \mathbb{R}$ be defined by:

$$A = \big\{\, x \in \mathbb{R} \mid x = -1 \text{ or } x = \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \,\big\} \tag{9}$$

In the subspace topology -1 is isolated, the set $\{\, -1 \,\}$ is open since:

$$A \cap \big(-\frac{3}{2},\, -\frac{1}{2}\big) = \{\, -1 \,\} \tag{10}$$

However, with the order induced from $\mathbb{R}$, the subspace order does not have $\{\, 1 \,\}$ as an open set. Any open set containing -1 must also contain some $\frac{1}{N}$, and hence also contain every $\frac{1}{n}$ for all $n > N$. ■

Example 1.3 (**Lexicographic Plane**) We can define a total order on $\mathbb{R}^2$. Given $(x_0,\, y_0)$ and $(x_1,\, y_1)$, define $(x_0,\, y_0) \leq (x_1,\, y_1)$ if and only if either $x_0 \leq x_1$ or, $x_0 = x_1$ and $y_0 \leq y_1$. That is, first examine the x axis and compare these. If they're identical, move on to the y axis. This is also called the *dictionary order* since it mimics how words are ordered in a dictionary. First, you compare the first letter, then the second, and so on. This order does **not** give the standard topology on $\mathbb{R}^2$, but it does give a good example to test ideas on. The lexicographic plane often serves as a counterexample to many plausible conjectures in topology. ■

2 The Long Line

This next example is a bit involved, but I hope you'll stick around for the ride. It is one of my favorite spaces. The axiom of choice tells us that the well-ordering theorem is true. That is, every set X has a well-order $\leq$ which is a total order such that every non-empty subset $A \subseteq X$ has a *smallest* element. There is only one well-order that I know of that arises naturally in mathematics, and that is the natural order on $\mathbb{N}$. $\mathbb{Z}$, with the standard order, is not well-ordered since $\mathbb{Z}$ has no least element (there is no *negative infinity* integer). What about $\mathbb{R}^+$, all positive numbers? Also no, since there is no smallest positive real number. How about $\mathbb{R}_{\geq 0}$, all positive numbers and zero? Also no. This set does have a smallest number, it is zero, but $\mathbb{R}^+ \subseteq \mathbb{R}_{\geq 0}$ is a non-empty subset that has no smallest element.

Well-orders are quite special. If $(X, \leq)$ is well-ordered and $x \in X$, either x is the largest element or there is a *next largest element*. The set $[x, \infty)$ is such that x is the smallest element. Removing it, considering (x, ∞), since x is not the largest element (well-ordered sets don't need to have a largest element, but it is possible for such an element to exist) this subset is non-empty, so there is a least element. This least element is the next largest element after x. So, knowing this, how could one possibly order the real numbers in a way that gives a well-order? Well, there's a reason the well-ordering theorem is equivalent to the axiom of choice, there's no constructive way to do it. But pretend, for a moment, that we accept the axiom of choice and the well-ordering theorem and let $\prec$ be a well-order on $\mathbb{R}$. Consider the sentence $P(x)$ *there are uncountably many elements less than* x. The reals are uncountable, so the set of real numbers satisfying this relation is non-empty. So there is a *least* element α. This is the *first* number that has uncountably many elements less than it. So every element $x \prec \alpha$ has only countably many elements less than x. This is bizarre. As noted, there is always a $+1$ element, a next largest element, in a well-order. α shows there does not need to be a *next smallest*, or a -1 before α.

Let ω be the set of all $x \in \mathbb{R}$ such that $x \prec \alpha$. ω is the *first-uncountable ordinal*. With the order $\prec$ restricted to ω, ω also becomes well-ordered. Let $X = \omega \times [0, 1)$, where $[0, 1)$ is the set of numbers between 0 and 1 with the **standard order**, including 0 but excluding 1. We can equip $\omega \times [0, 1)$ with the lexicographic ordering, saying $(x_0, y_0) \leq (x_1, y_1)$ if and only if $x_0 \preceq x_1$ or, $x_0 = x_1$ and $y_0 \leq y_1$. Equipping X with this order topology gives us the *long ray*. The real ray $\mathbb{R}_{\geq 0}$ can be thought of as stringing along countably many copies of $[0, 1)$ in a row. The long ray does this with *uncountably* many copies of $[0, 1)$. The long ray is extremely long. The long line is obtained by taking two copies of the long ray and gluing the endpoints together. Topologically, the long line is very pleasant and a nightmare. It has many very nice topological properties, but also serves as a great utensil for counterexamples to some very plausible claims.

3 Other Ordered Spaces

There are four more topologies a total order can give us.

Definition 3.1 (**Lower Limit Topology**) The lower limit topology on a totally ordered set $(X, \leq)$ is the topology τ generated by:

$$\mathcal{B} = \{\, [a,\, b) \subseteq X \mid a, b \in X \,\} \tag{11}$$

That is, the topology generated by half-open intervals closed on the left. ■

Definition 3.2 (**Upper Limit Topology**) The upper limit topology on a totally ordered set $(X, \leq)$ is the topology τ generated by:

$$\mathcal{B} = \{\, (a,\, b] \subseteq X \mid a, b \in X \,\} \tag{12}$$

That is, the topology generated by half-open intervals closed on the right. ■

Definition 3.3 (**Right Ray Topology**) The right ray topology on a totally ordered set $(X, \leq)$ is the topology τ generated by:

$$\mathcal{B} = \{\, (a,\, \infty) \subseteq X \mid a \in X \,\} \tag{13}$$

That is, the topology generated by rays that go off to the right. ■

Definition 3.4 (**Left Ray Topology**) The left ray topology on a totally ordered set $(X, \leq)$ is the topology τ generated by:

$$\mathcal{B} = \{\, (-\infty,\, a) \subseteq X \mid a \in X \,\} \tag{14}$$

That is, the topology generated by rays that go off to the left. ■

All of these give us plenty of examples of spaces, but we will be particularly concerned with the lower limit topology on $\mathbb{R}$. This space is so frequently discussed in counterexamples that it is given a name (it was the first known example of a normal spaces whose product is not normal. We'll get to this next lecture).

Definition 3.5 (**The Sorgenfrey Line**) The Sorgenfrey Line is the topological space $(\mathbb{R},\, \tau_S)$ where τ_S is the lower limit topology induced by the standard order $\leq$ on $\mathbb{R}$. ■

Recall that two topologies on a set X do not need to be comparable. The Sorgenfrey line, however, is comparable to the Euclidean line.

Theorem 3.1. *If $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$, and if τ_S is the Sorgenfrey topology, then $\tau_{\mathbb{R}} \subseteq \tau_S$.*

Proof. It suffices to show that basis elements $(a,\, b)$ in the standard topology are open in the Sorgenfrey line. Let $a < b$ and $r = \frac{b-a}{2}$. Define $\mathcal{U}_n$ to be:

$$\mathcal{U}_n = [a + \frac{r}{n+1},\, b) \tag{15}$$

Then $\mathcal{U}_n \in \tau_S$ since the Sorgenfrey topology is the lower limit topology on $\mathbb{R}$. But $\bigcup_n \mathcal{U}_n = (a,\, b)$, and the union of open sets is open, so $(a,\, b) \in \tau_S$. Hence $\tau_{\mathbb{R}} \subseteq \tau_S$. □

Theorem 3.2. *The Sorgenfrey line is Hausdorff.*

Proof. Since $(\mathbb{R}, \tau_{\mathbb{R}})$ is Hausdorff and $\tau_{\mathbb{R}} \subseteq \tau_S$, $(\mathbb{R}, \tau_S)$ is also Hausdorff. $\square$