Point-Set Topology: Lecture 20

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1 Connectedness

Connectedness is one of the fundamental notions in topology. Intuitively a connected space is one that is in *one piece*. It can be hard to make precise what one means by this, but it can be easier to describe what *disconnected* is. For intuition we use the plane. Two isolated discs in the plane should not be considered as a connected subspace since it is definitely not one piece (Fig. 1). We use this to motivate disconnected spaces.

Definition 1.1 (**Disconnected Topological Space**) A disconnected topological space is a topological space $(X,\,\tau)$ such that there are non-empty open subsets $\mathcal{U},\mathcal{V}\in\tau$ such that $\mathcal{U}\cap\mathcal{V}=\emptyset$ and $\mathcal{U}\cup\mathcal{V}=X$. ■

Example 1.1 The discrete topology on $\mathbb{Z}_2$ is disconnected. This space is two isolated points. To be precise, the set $\mathcal{U}=\{\,0\,\}$ is open and non-empty, the set $\mathcal{V}=\{\,1\,\}$ is open and non-empty, and these two sets satisfy $\mathcal{U}\cap\mathcal{V}=\emptyset$ and $\mathcal{U}\cup\mathcal{V}=\mathbb{Z}_2$. ■

Example 1.2 If X is any set containing at least two points, and if τ is the discrete topology, then $(X,\,\tau)$ is disconnected. Let $x\in X$ be one point and define $\mathcal{U}=\{\,x\,\}$. Since τ is the discrete topology $\mathcal{U}$ is open and non-empty. Let $\mathcal{V}=X\setminus\mathcal{U}$. Again, since τ is the discrete topology, $\mathcal{V}$ is open and since X

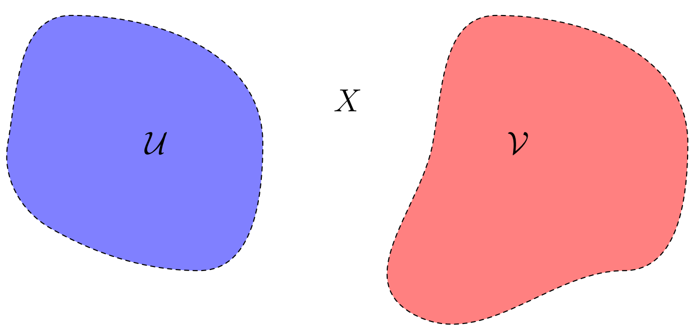

Figure 1: A Disconnected Topological Space

has at least two points it is also non-empty. But then $\mathcal{U}$ and $\mathcal{V}$ are non-empty open subsets such that $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{U} \cup \mathcal{V} = X$, showing that (X, τ) is disconnected. ■

Connected is just *not disconnected.*

Definition 1.2 (**Connected Topological Space**) A connected topological space is a topological space (X, τ) that is not disconnected. ■

Some familiar spaces like $\mathbb{R}$ and $\mathbb{R}^2$ are connected, but it takes a bit of work to show this. The spaces that are easy to show are connected straight from the definition have less-than-useful topologies.

Example 1.3 If X is any set and τ is the indiscrete topology, then (X, τ) is connected. There are no two disjoint open sets $\mathcal{U}, \mathcal{V}$ that are non-empty and cover X since the only open sets are $\emptyset$ and X. So (X, τ) is connected. ■

Example 1.4 The particular point topology on $\mathbb{R}$ defines a set $\mathcal{U}$ to be open if and only if $0 \in \mathcal{U}$ or $\mathcal{U} = \emptyset$. Hence any two non-empty open sets that cover $\mathbb{R}$ must have 0 in common, meaning we cannot separate the space into two disjoint non-empty open sets, so the particular point space is connected. Intuitively, every point is *connected* to zero. ■

Example 1.5 The excluded point topology on $\mathbb{R}$ defines a set $\mathcal{U}$ to be open if and only if $0 \notin \mathcal{U}$ or $\mathcal{U} = \mathbb{R}$. Because of this if $\mathcal{U}$ and $\mathcal{V}$ are open sets that cover $\mathbb{R}$, one of these sets must be $\mathbb{R}$. So it is impossible to separate the space using disjoint non-empty open sets. ■

Example 1.6 The finite complement topology on $\mathbb{R}$ is connected. Given any non-empty open subsets $\mathcal{U}, \mathcal{V}$, the intersection can not be empty since $\mathbb{R} \setminus \mathcal{U}$ and $\mathbb{R} \setminus \mathcal{V}$ are both finite, meaning $\mathcal{U} \cap \mathcal{V}$ is infinite (since $\mathbb{R}$ is infinite). So the finite complement topology on $\mathbb{R}$ is connected. ■

Example 1.7 For similar reasons, the countable complement topology on $\mathbb{R}$ yields a connected space. Any two non-empty open subsets must have non-empty intersection since $\mathbb{R}$ is uncountable and the complements of two non-empty open subsets is countable (and hence so is the union of their complements). ■

Example 1.8 The rationals $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$ are disconnected. Let $\mathcal{U}$ be all positive rational numbers x such that $x^2 > 2$. Let $\mathcal{V}$ be all rational numbers x such that either $x < 0$ or $x^2 < 2$. There is no rational number whose square is 2, so $\mathcal{U}$ and $\mathcal{V}$ are non-empty disjoint open subsets whose union is the entirety of $\mathbb{Q}$. So the rationals are disconnected. ■

Example 1.9 Let $X \subseteq \mathbb{R}$ be defined by $X = (-\infty,\, 0) \cup (0,\, \infty)$. This is the real line with the origin removed. Equipping this with the subspace topology yields a disconnected space. Setting $\mathcal{U} = (-\infty,\, 0)$ and $\mathcal{V} = (0,\, \infty)$ shows why. ■

When first discussing open and closed sets many students have trouble realizing that *open* does not mean *not closed*, and *closed* does not mean *not open*. It is possible for a subset to be open and not closed, closed and not open, neither open nor closed, and both open and closed. This last part is particularly hard to grasp since $\mathbb{R}$ has no subsets $\mathcal{U} \subseteq \mathbb{R}$ that are both open and closed with the exception of $\mathcal{U} = \mathbb{R}$ and $\mathcal{U} = \emptyset$. This is because the real line is *connected* and connected spaces have no proper non-empty subsets that are both open and closed. Let's prove this.

Theorem 1.1. *If (X, τ) is a topological space, then it is disconnected if and only if there is a non-empty open proper subset $\mathcal{U} \subsetneq X$ that is also closed.*

Proof. If (X, τ) is disconnected there exists non-empty disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ whose union is X. But then $X \setminus \mathcal{U} = \mathcal{V}$, and $\mathcal{V}$ is open, so $\mathcal{U}$ is closed. But $\mathcal{U}$ is also open, so $\mathcal{U}$ is a non-empty proper subset of X that is also closed. Now suppose there is a proper subset $\mathcal{U} \subsetneq X$ that is non-empty and both open and closed. Since $\mathcal{U}$ is closed, $\mathcal{V} = X \setminus \mathcal{U}$ is open. But since $\mathcal{U}$ is proper, $\mathcal{V}$ is non-empty. But then $\mathcal{U}$ and $\mathcal{V}$ are disjoint non-empty open subsets whose union is X, so (X, τ) is disconnected. □

Theorem 1.2. *If (X, τ) is a topological space, then it is disconnected if and only if there are non-empty disjoint closed subsets $\mathcal{C}, \mathcal{D} \subseteq X$ such that $\mathcal{C} \cup \mathcal{D} = X$.*

Proof. If (X, τ) is disconnected there are non-empty disjoint open subsets $\mathcal{U}, \mathcal{V}$ such that $\mathcal{U} \cup \mathcal{V} = X$. But then $\mathcal{C} = X \setminus \mathcal{U}$ and $\mathcal{D} = X \setminus \mathcal{V}$ are closed non-empty disjoint subsets whose union is X. Now, suppose there are non-empty disjoint closed subsets $\mathcal{C}, \mathcal{D} \subseteq X$ such that $\mathcal{C} \cup \mathcal{D} = X$. But then $\mathcal{U} = X \setminus \mathcal{C}$ and $\mathcal{V} = X \setminus \mathcal{D}$ are open non-empty disjoint sets whose union is X, so (X, τ) is disconnected. □

One of the most useful theorems of connected spaces is that the continuous image of a connected topological space is still connected. To prove this requires a small theorem about subspaces.

Theorem 1.3. *If (X, τ_X) and (Y, τ_Y) are topological spaces, if $f : X \to Y$ is continuous, and if $\mathcal{U} \subseteq f[X]$ is open in the subspace topology $\tau_{Y_{f[X]}}$, then $f^{-1}[\mathcal{U}]$ is open.*

Proof. Since $\mathcal{U} \in \tau_{Y_{f[X]}}$, by the definition of the subspace topology there is $\tilde{\mathcal{U}} \in \tau_Y$ such that $\mathcal{U} = f[X] \cap \tilde{\mathcal{U}}$. But then we have:

$$\begin{aligned} f^{-1}[\mathcal{U}] &= f^{-1}\big[f[X] \cap \tilde{\mathcal{U}}\big] && (1)\\ &= f^{-1}\big[f[X]\big] \cap f^{-1}[\tilde{\mathcal{U}}] && (2)\\ &= X \cap f^{-1}[\tilde{\mathcal{U}}] && (3)\\ &= f^{-1}[\tilde{\mathcal{U}}] && (4) \end{aligned}$$

but f is continuous, so $f^{-1}[\tilde{\mathcal{U}}]$ is open. Hence $f^{-1}[\mathcal{U}]$ is open. □

Theorem 1.4. *If (X, τ_X) is a connected topological space, if (Y, τ_Y) is a topological space, and if $f : X \to Y$ is a continuous function, then $(f[X], \tau_{Y_{f[X]}})$ is connected where $\tau_{Y_{f[A]}}$ is the subspace topology from τ_Y.*

Proof. Suppose not. Then there are disjoint non-empty open subsets $\mathcal{U}, \mathcal{V} \in \tau_{Y_{f[X]}}$ such that $\mathcal{U} \cup \mathcal{V} = f[X]$. But then, since $\tau_{Y_{f[X]}}$ is the subspace topology and $f : X \to Y$ is continuous, $f^{-1}[\mathcal{U}]$ and $f^{-1}[\mathcal{V}]$ are open non-empty subsets of X. But:

$$f^{-1}[\mathcal{U}] \cup f^{-1}[\mathcal{V}] = f^{-1}[\mathcal{U} \cup \mathcal{V}] = f^{-1}\big[f[X]\big] = X \tag{5}$$

so $f^{-1}[\mathcal{U}]$ and $f^{-1}[\mathcal{V}]$ separate X, but (X, τ) is connected, a contradiction. Hence, $(f[X], \tau_{Y_{f[X]}})$ is connected. □

This does not say that (Y, τ_Y) is connected, only that the image of a connected space (X, τ_X) remains connected. If the function f happens to be surjective, then we can upgrade this theorem.

Theorem 1.5. *If (X, τ_X) is a connected topological space, if (Y, τ_Y) is a topological space, and if $f : X \to Y$ is a surjective continuous function, then (Y, τ_Y) is connected.*

Proof. By the previous theorem $(f[X], \tau_{Y_{f[X]}})$ is connected. But f is surjective so $f[X] = Y$. And the subspace topology of Y with respect to τ_Y is just τ_Y. That is, $\tau_{Y_{f[X]}} = \tau_Y$. So (Y, τ_Y) is connected. □

Theorem 1.6. *If (X, τ_X) is a connected topological space, if (Y, τ_Y) is a topological space, and if $f : X \to Y$ is a quotient map, then (Y, τ_Y) is connected.*

Proof. Quotient maps are continuous and surjective, so by the previous theorem (Y, τ_Y) is connected. □

Theorem 1.7. *If (X, τ) is a connected topological space, and if R is an equivalence relation on X, then $(X/R, \tau_{X/R})$ is connected where $\tau_{X/R}$ is the quotient topology.*

Proof. The canonical quotient function $q : X \to X/R$ defined by $q(x) = [x]$ is a quotient map. By the previous theorem, since (X, τ) is connected, so is $(X/R, \tau_{X/R})$. □

Connectedness is one of the few properties that quotient maps preserve. Remember that quotients do not need to preserve the Hausdorff condition, first or second countability, or any separation properties. The three main things they preserve are *sequentialness*, *connectedness*, and *compactness*.

Just like how disconnected has a few equivalent definitions, so does connected.

Theorem 1.8. *If (X, τ) is a topological space, then it is connected if and only if the only subsets of X with empty topological boundary are $\emptyset$ and X.*

Proof. For suppose $(X,\,\tau)$ is disconnected. Then there is a non-empty proper subset $A\subsetneq X$ such that A is open and closed. But since A is closed, $\mathrm{Cl}_\tau(A)=A$. But since A is open, $\mathrm{Int}_\tau(A)=A$. But then:

$$\partial_\tau(A)=\mathrm{Cl}_\tau(A)\setminus\mathrm{Int}_\tau(A)=A\setminus A=\emptyset \tag{6}$$

Going the other way, suppose there is a non-empty proper subset $A\subsetneq X$ with empty boundary, $\partial_\tau(A)=\emptyset$. But $\mathrm{Int}_\tau(A)\subseteq A$ for all $A\subseteq X$ and $A\subseteq\mathrm{Cl}_\tau(A)$ as well. Hence $\mathrm{Int}_\tau(A)\subseteq\mathrm{Cl}_\tau(A)$. But if $\partial_\tau(A)=\emptyset$ then $\mathrm{Cl}_\tau(A)\setminus\mathrm{Int}_\tau(A)=\emptyset$, and thus $\mathrm{Cl}_\tau(A)\subseteq\mathrm{Int}_\tau(A)$. But then $A=\mathrm{Cl}_\tau(A)=\mathrm{Int}_\tau(A)$, so A is both closed and open. But A is non-empty and a proper subset of X, and therefore $(X,\,\tau)$ is disconnected. So $(X,\,\tau)$ is connected if and only if the only subsets of X with empty boundary are $\emptyset$ and X. □

Theorem 1.9. *If $(X,\,\tau)$ is a topological space, and if $\big(\mathbb{Z}_2,\,\mathcal{P}(\mathbb{Z}_2)\big)$ is the discrete topological space on $\mathbb{Z}_2$, then $(X,\,\tau)$ is connected if and only if for every continuous function $f:X\rightarrow\mathbb{Z}_2$ it is true that f is constant.*

Proof. Suppose $(X,\,\tau)$ is connected and $f:X\rightarrow\mathbb{Z}_2$ is not constant. Then there is an $x\in X$ such that $f(x)=0$ and a $y\in X$ with $f(y)=1$. But then $f^{-1}[\{\,0\,\}]$ and $f^{-1}[\{\,1\,\}]$ are non-empty disjoint open sets that cover X which is a contradiction since $(X,\,\tau)$ is connected. So f must be a constant. Conversely, suppose every continuous function $f:X\rightarrow\mathbb{Z}_2$ is a constant and suppose $(X,\,\tau)$ is disconnected. Then there is a non-empty proper subset $A\subsetneq X$ that is both open and closed. Define $f:X\rightarrow\mathbb{Z}_2$ via:

$$f(x)=\begin{cases}0 & x\in A\\ 1 & x\notin A\end{cases} \tag{7}$$

Since A is non-empty there is an $x\in X$ such that $f(x)=0$ and since A is also a proper subset $X\setminus A$ is non-empty so there is a $y\in X$ such that $f(y)=1$. That is, f is not a constant function. But f is continuous. The pre-image of $\{\,0\,\}$ is A, which is open, and the pre-image of $\{\,1\,\}$ is $X\setminus A$ which is also open since A is closed. But by hypothesis the only continuous functions from X to $\mathbb{Z}_2$ are constants, a contradiction. Therefore $(X,\,\tau)$ is connected. □

2 The Connected Subsets of the Real Line

Now we classify all subsets of the real line and use this to prove the intermediate value theorem, one of the fundamental results of real analysis and calculus, using just a little bit of topology.

Theorem 2.1. *If $\tau_{\mathbb{R}}$ is the standard Euclidean topology on $\mathbb{R}$, then $(\mathbb{R},\,\tau_{\mathbb{R}})$ is connected.*

Proof. Suppose not and let $\mathcal{U}$ and $\mathcal{V}$ be non-empty disjoint open sets with $\mathcal{U}\cup\mathcal{V}=\mathbb{R}$. Since $\mathcal{U}$ and $\mathcal{V}$ cover $\mathbb{R}$ and are disjoint, either $0\in\mathcal{U}$ or $0\in\mathcal{V}$, but not both. Suppose $0 \in \mathcal{U}$ (the idea is symmetric). Since $\mathcal{V}$ is non-empty there is some $x \in \mathcal{V}$. Either $x < 0$ or $0 < x$ by trichotomy. Suppose $0 < x$ (again, the proof is symmetric). Let $E \subseteq \mathbb{R}$ be defined by:

$$E = \{\, y \in \mathcal{U} \mid 0 < y \text{ and } y < x \,\} \tag{8}$$

Then E is bounded above by x and hence there is a least upper bound $c \in \mathbb{R}$. Since $\mathcal{U}$ and $\mathcal{V}$ cover $\mathbb{R}$ either $c \in \mathcal{U}$ or $c \in \mathcal{V}$. Suppose $c \in \mathcal{U}$. Since $\mathcal{U}$ is open there is an $\varepsilon > 0$ such that $|y - c| < \varepsilon$ implies $y \in \mathcal{U}$. But then $c + \varepsilon/2$ is an element of $\mathcal{U}$ that is still bounded by x since $x \in \mathcal{V}$ and hence $|x - c| \geq \varepsilon$. But then $c + \varepsilon/2$ is an element of E that is greater than c, a contradiction since c is the least upper bound of E. So $c \notin \mathcal{U}$. But then $c \in \mathcal{V}$. But $\mathcal{V}$ is open so there is an $\varepsilon > 0$ such that $|y - c| < \varepsilon$ implies $y \in \mathcal{V}$. But then $c - \varepsilon/2$ is an upper bound for E that is less than c, a contradiction since c is the least upper bound of E. So $c \notin \mathcal{V}$. But $\mathcal{U}$ and $\mathcal{V}$ cover $\mathbb{R}$, which is a contradiction. Hence $(\mathbb{R}, \tau_{\mathbb{R}})$ is connected. □

Theorem 2.2. *If $A \subseteq \mathbb{R}$ is a connected subset with respect to the standard topology $\tau_{\mathbb{R}}$, then A is one of the following sets:*

$$A = \begin{cases} (-\infty,\, a) \\ (-\infty,\, a] \\ [a,\, \infty) \\ (a,\, \infty) \\ (a,\, b) \\ [a,\, b) \\ (a,\, b] \\ [a,\, b] \\ \mathbb{R} \\ \emptyset \end{cases} \tag{9}$$

for some $a, b \in \mathbb{R}$.

Proof. Let $a, b \in \mathbb{R} \cup \{\, \pm\infty \,\}$ be defined by $a = \inf(A)$ and $b = \sup(A)$. Apply the same argument as before with the set of all real numbers between a and b. Complete the proof by asking if a and b are finite and whether or not $a, b \in A$. This will give the table of possibilities above. □

Theorem 2.3 (**Intermediate Value Theorem**). *If $a, b \in \mathbb{R}$, $a < b$, and if $f : [a,\, b] \to \mathbb{R}$ is continuous with respect to the standard topologies on $[a,\, b]$ and $\mathbb{R}$, then for all real numbers d between $f(a)$ and $f(b)$ there is a $c \in (a,\, b)$ such that $f(c) = d$.*

Proof. Since $[a,\, b]$ is connected and f is continuous, $f\big[[a,\, b]\big]$ is connected as well. But connected non-empty subsets of the real line are intervals (open, closed, half-open, or infinite). Meaning if $f(a) \leq f(b)$, then $[f(a),\, f(b)] \subseteq f\big[[a,\, b]\big]$ and if $f(b) \leq f(a)$, then $[f(b),\, f(a)] \subseteq f\big[[a,\, b]\big]$. Either way, for all d between $f(a)$ and $f(b)$ there is a c between a and b such that $f(c) = d$. □

Theorem 2.4 (**1D Borsuk-Ulam Theorem**). *If $f : \mathbb{S}^1 \to \mathbb{R}$ is a continuous function, then there is a point $\mathbf{x} \in \mathbb{S}^1$ such that $f(\mathbf{x}) = f(-\mathbf{x})$. That is, there are two antepodal points on the circle that map to the same real number.*

Proof. Define $g : \mathbb{R} \to \mathbb{S}^1$ via $g(t) = \big(\cos(2\pi t), \sin(2\pi t)\big)$ and $h : [0, 1] \to \mathbb{R}$ via $h(t) = f\big(g(t)\big) - f\big(g(t + \pi)\big)$. If $h(0) = 0$, we are done since $(1, 0)$ and $(-1, 0)$ are then points with $f\big((1, 0)\big) = f\big((-1, 0)\big)$. Suppose $h(0)$ is positive (the idea is symmetric), $h(t) = c > 0$. Then:

$$
\begin{aligned}
h\big(\frac{1}{2}\big) &= f\Big(\big(\cos(\pi), \sin(\pi)\big)\Big) - f\Big(\big(\cos(0), \sin(0)\big)\Big) && (10)\\
&= f\big((-1, 0)\big) - f\big((1, 0)\big) && (11)\\
&= -\Big(f\big((1, 0)\big) - f\big((-1, 0)\big)\Big) && (12)\\
&= -h(0) && (13)\\
&= -c && (14)
\end{aligned}
$$

and $-c < 0$. So, by the intermediate value theorem, there is some real number $0 < t_0 < \frac{1}{2}$ such that $h(t_0) = 0$. But then $f\big(g(t_0)\big) = f\big(g(t_0 + \pi)\big)$, and hence $g(t_0)$ and $g(t_0 + \pi)$ are antepodal points that are mapped to the same real number via f. □

This is called the 1D Borsuk-Ulam theorem because the actual Borsuk-Ulam theorem is quite stronger. If $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous, then there are antepodal points $\mathbf{x}, -\mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.

3 Path-Connectedness

The more intuitive notion of connected, the idea one probably thinks of when hearing *connected*, is being able to walk from one point to another while staying in the space. This is a stronger notion and is called *path connected*.

Definition 3.1 (**Path Connected Topological Space**) A path connected topological space is a topological space (X, τ) such that for all $x, y \in X$ there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$, where $[0, 1]$ has the subspace topology from $\mathbb{R}$. ■

Theorem 3.1. *If (X, τ) is path connected, then it is connected.*

Proof. Suppose not. Then there are non-empty disjoint open subsets $\mathcal{U}, \mathcal{V}$ such that $\mathcal{U} \cup \mathcal{V} = X$. But since $\mathcal{U}$ and $\mathcal{V}$ are non-empty, there are $x \in \mathcal{U}$ and $y \in \mathcal{V}$. But (X, τ) is path connected so there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. But then $f^{-1}[\mathcal{U}]$ and $f^{-1}[\mathcal{V}]$ are non-empty disjoint open subsets that cover $[0, 1]$. But $[0, 1]$ is connected, a contradiction. Hence (X, τ) is connected. □

Theorem 3.2. *If $n \in \mathbb{N}$ and $\tau_{\mathbb{R}^n}$ is the Euclidean topology, then $(\mathbb{R}^n, \tau_{\mathbb{R}^n})$ is path connected.*

Proof. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define $\alpha : [0,\, 1] \to \mathbb{R}^n$ by $\alpha(t) = (1-t)\mathbf{x} + t\mathbf{y}$. Then α is continuous, $\alpha(0) = \mathbf{x}$, and $\alpha(1) = \mathbf{y}$. Hence, $(\mathbb{R}^n,\, \tau_{\mathbb{R}^n})$ is path connected. □

Theorem 3.3. *If $(X,\, \tau_X)$ is a path connected topological space, if $(Y,\, \tau_Y)$ is a topological space, and if $f : X \to Y$ is continuous, then $(f[X],\, \tau_{Y_{f[X]}})$ is path connected.*

Proof. Let $y_0, y_1 \in f[X]$. Then there are $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. But $(X,\, \tau_X)$ is path connected, so there is a continuous function $\alpha : [0,\, 1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. But then $f \circ \alpha : [0,\, 1] \to f[X]$ is the composition of continuous functions, which is hence continuous, such that $(f \circ \alpha)(0) = f(x_0) = y_0$ and $(f \circ \alpha)(1) = f(x_1) = y_1$. So $(f[X],\, \tau_{Y_{f[X]}})$ is path connected. □

Theorem 3.4. *If $(X,\, \tau)$ is a topological space, if $A, B \subseteq X$ are such that $(A,\, \tau_A)$ and $(B,\, \tau_B)$ are path connected, where τ_A and τ_B are the subspace topologies, and if $A \cap B \neq \emptyset$, then $(A \cup B,\, \tau_{A\cup B})$ is path connected.*

Proof. Since $A \cap B \neq \emptyset$ there is an $x \in A \cap B$. Let $y_0, y_1 \in A \cup B$. Since $y_0 \in A \cup B$, either $y_0 \in A$ or $y_0 \in B$. Either way, since $x \in A \cap B$ and $(A,\, \tau_A)$ and $(B,\, \tau_B)$ are path connected, there is a continuous function $f : [0,\, 1] \to A \cup B$ such that $f(0) = y_0$ and $f(1) = x$. Similarly there is a continuous function $g : [0,\, 1] \to A \cup B$ such that $g(0) = x$ and $g(1) = y_1$. By the pasting lemma the function $h : [0,\, 1] \to A \cup B$ defined by:

$$h(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{15}$$

is continuous. But then $h(0) = f(0) = y_0$ and $h(1) = g(1) = y_1$, so $h : [0,\, 1] \to A \cup B$ is a continuous function such that $h(0) = y_0$ and $h(1) = y_1$. So $(A \cup B,\, \tau_{A\cup B})$ is path connected. □

Connected need not imply path connected.

Example 3.1 (**The Infinite Broom**) The infinite broom is a subset of $\mathbb{R}^2$ defined by taking all closed line segments from $(0,\, 0)$ to $(1,\, \frac{1}{n+1})$ for all $n \in \mathbb{N}$, together with the half-open line segment between $(\frac{1}{2},\, 0)$ and $(1,\, 0)$, including $(1,\, 0)$ but excluding $(\frac{1}{2},\, 0)$. See Fig. 2. If we did not include this last half-open interval, the space would be path connected. But with this half-open interval the space is connected but not path connected. There is no path from $(0,\, 0)$ to $(1,\, 0)$. However, since the infinite broom has the subspace topology from $\mathbb{R}^2$, any open subset that contains the half-open interval must contain points from infinitely many of the line segments from $(0,\, 0)$ to $(1,\, \frac{1}{n+1})$. Because of this it is impossible to disconnect the space with two non-empty disjoint open subsets, showing that the infinite broom is connected. ■

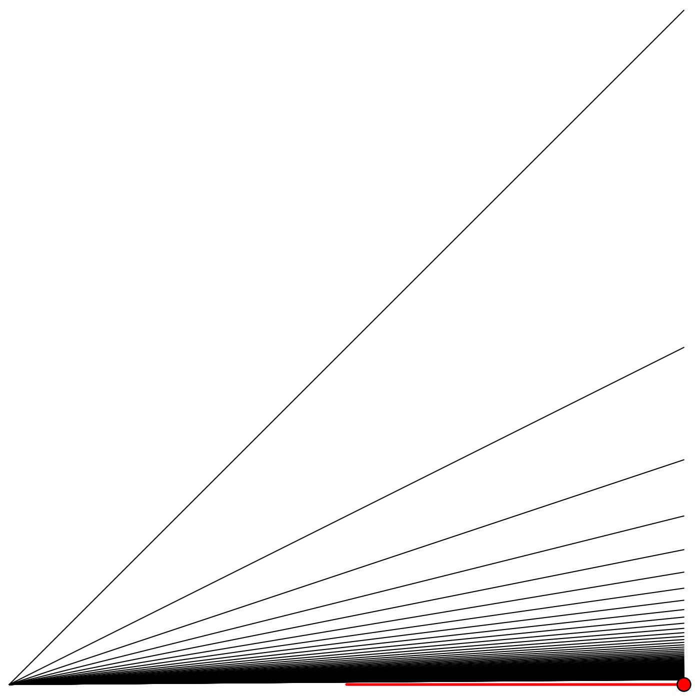

Figure 2: The Infinite Broom

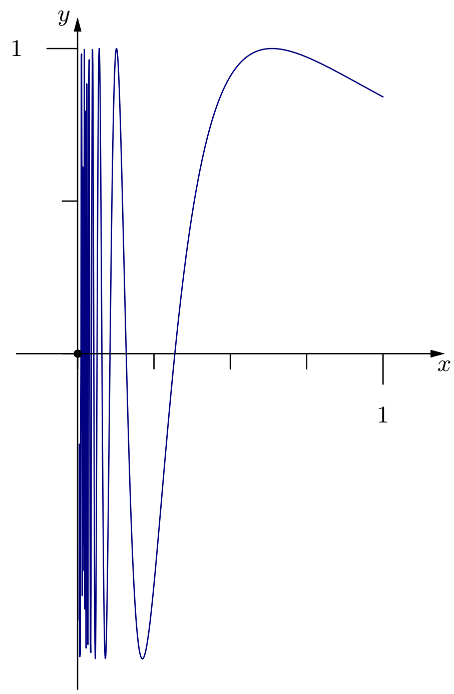

Figure 3: The Topologist's Sine Curve

Example 3.2 (**The Topologist's Sine Curve**) The topologist's sine curve is a subset of $\mathbb{R}^2$ defined by taking all points of the form $\big(x, \sin(1/x)\big)$ with $x \in (0, 1]$ and adding the origin $(0, 0)$ (Fig. 3). For reasons similar to the infinite broom, the topologist's sine curve is connected but not path connected. ■