Point-Set Topology: Lecture 24

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1 Paracompact Spaces

Paracompactness is an idea that seems, at first glance, strange, and is nowhere near as strong as compactness. One may then wonder why on Earth it deserves study, let alone a name. One of the motivations is the Urysohn metrization theorem. This theorem goes one way, a second countable regular Hausdorff space is metrizable. It does not reverse. The discrete topology $\mathbb{R}$ is metrizable but not second countable. It would be nice to have a theorem that gives necessary and sufficient conditions for a space to metrizable. The Nagata-Smirnov and Smirnov metrization theorems do this. At the heart of both theorems is the idea of local finiteness. The Nagata-Smirnov theorem requires σ locally finite bases, the Smirnov theorem uses paracompactness. We take the time to develop these and similar ideas. This leads in to the Stone paracompactness theorem and these two metrization theorems.

Definition 1.1 (**Locally Finite Collection**) A locally finite collection in a topological space $(X,\,\tau)$ is a subset $\mathcal{A}\subseteq\mathcal{P}(X)$ such that for all $x\in X$ there is a $\mathcal{U}\in\tau$ such that $x\in\mathcal{U}$ and only finitely many $A\in\mathcal{A}$ are such that $A\cap\mathcal{U}\neq\emptyset$. ∎

Note there is no requirement that $\mathcal{A}$ consist of open sets or closed sets. There is no requirement that $\mathcal{A}$ covers the space either. All that is required is local finiteness.

Definition 1.2 (**Refinement of a Collection**) A refinement of a collection $\mathcal{A}\subseteq\mathcal{P}(X)$ in a topological space $(X,\,\tau)$ is a set $\tilde{\mathcal{A}}\subseteq\mathcal{P}(X)$ such that for all $\tilde{A}\in\tilde{\mathcal{A}}$ there is an $A\in\mathcal{A}$ such that $\tilde{A}\subseteq A$. ∎

The idea behind a refinement is that we take sets in $\mathcal{A}$ and *shrink* them, in some sense. Again, in general, there is no requirement that $\mathcal{A}$ or $\tilde{\mathcal{A}}$ consist of open or closed sets. *Open refinements* are refinements consisting of open sets, and *closed refinements* consist of closed sets. This idea is used to define paracompactness.

Definition 1.3 (**Paracompact Topological Space**) A paracompact topological space is a topological space $(X,\,\tau)$ such that for all open covers $\mathcal{O}\subseteq\tau$ of X there exists a locally finite open refinement $\mathcal{X}\subseteq\tau$ of $\mathcal{O}$ that is an open cover of $(X,\,\tau)$. ∎

This is a very weak notion, many familiar spaces are paracompact, and yet it has enormous use in manifold theory and the study of metric spaces. In particular because every manifold and every metrizable space is paracompact.

Theorem 1.1. *If $(X,\,\tau)$ is a compact topological space, then it is paracompact.*

Proof. In a compact space every open cover has a finite subcover, which is certainly a locally finite open refinement of the cover. □

Far weaker than compactness, σ compact plus locally compact Hausdorff implies paracompact. We've discussed local compactness in metric spaces, and the definition has very little difference in topological spaces.

Definition 1.4 (**Locally Compact Topological Space**) A locally compact topological space is a topological space $(X,\,\tau)$ such that for all $x\in X$ there is an open set $\mathcal{U}\in\tau$ and a compact set $K\subseteq X$ such that $x\in\mathcal{U}$ and $\mathcal{U}\subseteq K$. ■

Before proving locally compact σ compact spaces are paracompact, we'll need a little lemma.

Theorem 1.2. *If $(X,\,\tau)$ is locally compact and Hausdorff and if $x\in X$, then there is a $\mathcal{U}\in\tau$ such that $x\in\mathcal{U}$ and $\mathrm{Cl}_\tau(\mathcal{U})$ is compact.*

Proof. Since $(X,\,\tau)$ is locally compact there is a $\mathcal{U}\in\tau$ and a compact $K\subseteq X$ such that $x\in\mathcal{U}$ and $\mathcal{U}\subseteq K$. But $(X,\,\tau)$ is Hausdorff, so K is closed. But then $\mathrm{Cl}_\tau(\mathcal{U})\subseteq K$. But then $\mathrm{Cl}_\tau(\mathcal{U})$ is a closed subset of a compact space, which is therefore compact. □

Theorem 1.3. *If $(X,\,\tau)$ is σ compact, locally compact, and Hausdorff, then it is compactly exhaustible.*

Proof. Since $(X,\,\tau)$ is σ compact there are countably many sets $\mathcal{C}_n$, $n\in\mathbb{N}$, each of which is compact and such that they cover the space. For all $n\in\mathbb{N}$ and for all $x\in\mathcal{C}_n$, since $(X,\,\tau)$ is locally compact and Hausdorff, there is a $\mathcal{V}_{x,n}\in\tau$ such that $x\in\mathcal{V}_{x,n}$ and $\mathrm{Cl}_\tau(\mathcal{V}_{x,n})$ is compact. But these sets cover $\mathcal{C}_n$, which is compact, so we can do it with finitely many, $\mathcal{V}_{0,n},\,\dots,\,\mathcal{V}_{N,n}$. Since this is a finite collection we have:

$$\mathrm{Cl}_\tau\Big(\bigcup_{k=0}^{N}\mathcal{V}_{k,n}\Big)=\bigcup_{k=0}^{N}\mathrm{Cl}_\tau(\mathcal{V}_{k,n}) \tag{1}$$

Define $\mathcal{U}_n$ via:

$$\mathcal{U}_n=\bigcup_{k=0}^{N}\mathcal{V}_{k,n} \tag{2}$$

Then $\mathcal{U}_n$ is open and by the previous equation $\mathrm{Cl}_n(\mathcal{U}_n)$ is the finite union of compact sets, which is therefore compact. Recursively define $\mathcal{W}_n$ as follows. Set $\mathcal{W}_0=\mathcal{U}_0$. Let $\mathcal{W}_n\in\tau$ be such that $\bigcup_{k=0}^{n}\mathcal{U}_k\subseteq\mathcal{W}_n$, $\mathcal{W}_{n-1}\subseteq\mathcal{W}_n$, and such that $\mathrm{Cl}_\tau(\mathcal{W}_n)$ is compact. Define $\mathcal{W}_{n+1}$ as follows. Since $\mathrm{Cl}_\tau(\mathcal{W}_n)$ and $\mathrm{Cl}_\tau(\mathcal{U}_{n+1})$

are compact, so is the union. Thus, by the previous argument, we can cover it in finitely many open sets $\mathcal{V}_0, \dots, \mathcal{V}_N$, each of which has compact closure. Define:

$$\mathcal{W}_{n+1} = \bigcup_{k=0}^{N} \mathcal{V}_k \tag{3}$$

Then $\mathcal{W}_{n+1}$ is open and:

$$\mathrm{Cl}_\tau(\mathcal{W}_{n+1}) = \mathrm{Cl}_\tau\Big(\bigcup_{k=0}^{N} \mathcal{V}_k\Big) = \bigcup_{k=0}^{N} \mathrm{Cl}_\tau(\mathcal{V}_k) \tag{4}$$

which is the finite union of compact sets, so it is compact. But moreover, from the construction, since the $\mathcal{V}_k$ cover $\mathrm{Cl}_\tau(\mathcal{W}_n)$, we have $\mathrm{Cl}_\tau(\mathcal{W}_n) \subseteq \mathcal{W}_{n+1}$. Define:

$$K_n = \mathrm{Cl}_\tau(\mathcal{W}_n) \tag{5}$$

Then K_n is compact and $K_n \subseteq \mathrm{Int}_\tau(K_{n+1})$ since $\mathcal{W}_{n+1} \subseteq \mathrm{Int}_\tau(K_{n+1})$. Morever $\bigcup_{n\in\mathbb{N}} K_n = X$ since $\mathcal{C}_n \subseteq \mathcal{U}_n$, $\mathcal{U}_n \subseteq \mathcal{W}_n$, and $\mathcal{W}_n \subseteq K_n$. Since the $\mathcal{C}_n$ cover X, so do the K_n. Hence, (X, τ) is compactly exhaustible. □

Theorem 1.4. *If (X, τ) is compactly exhaustible and Hausdorff, then it is paracompact.*

Proof. Let $K : \mathbb{N} \to \mathcal{P}(X)$ be such that for all $n \in \mathbb{N}$ K_n is compact, $K_n \subseteq \mathrm{Int}_\tau(K_{n+1})$, and $\bigcup_{n\in\mathbb{N}} K_n = X$. Note that, since $\mathrm{Int}_\tau(K_n) \subseteq K_{n+1}$ is open, and K_{n+1} is compact, $K_{n+1} \setminus \mathrm{Int}_\tau(K_n)$ is compact. Let $\mathcal{O}$ be an open cover. We must find a locally finite open refinement $\mathcal{X}$ of $\mathcal{O}$. But $\mathcal{O}$ covers X, so it covers $K_{n+1} \setminus \mathrm{Int}_\tau(K_n)$. By compactness there are finitely many $\mathcal{V}_0, \dots, \mathcal{V}_{n_N}$ that cover $K_{n+1} \setminus \mathrm{Int}_\tau(K_n)$. Define Δ_n via:

$$\Delta_n = \Big\{\, \mathcal{V}_k \cap \big(\mathrm{Int}_\tau(K_{n+2}) \setminus K_{n-1}\big) \mid 0 \leq k \leq N_n \Big\} \tag{6}$$

(define $K_{-1} = \emptyset$ for the case $n = 0$). But (X, τ) is Hausdorff, so each K_n is closed, hence $\mathrm{Int}_\tau(K_{n+2}) \setminus K_{n-1}$ is open, meaning all elements of Δ are open. The set $\mathcal{X} = \bigcup_{n\in\mathbb{N}} \Delta_n$ is a locally finite open refinement. It is an open refinement, the elements are open and are contained as subsets of the elements of $\mathcal{O}$ by construction. It is also an open cover since it covers each K_n, and the K_n cover X. Lastly, it is locally finite. Every element of X is contained in some $\mathrm{Int}_\tau(K_{n+1}) \setminus K_{n-1}$ for some n, and $\mathcal{X}$ has only finitely many elements with non-empty intersection with this set, the elements of Δ_n. So $\mathcal{X}$ is a locally finite open refinement of $\mathcal{O}$ that covers X, so (X, τ) is paracompact. □

Theorem 1.5. *If (X, τ) is paracompact, and if $\mathcal{C} \subseteq X$ is closed, then $(\mathcal{C}, \tau_{\mathcal{C}})$ is paracompact where $\tau_{\mathcal{C}}$ is the subspace topology.*

Proof. The proof is a mimicry of the idea for compact spaces. Given an open cover $\mathcal{O}$ of $\mathcal{C}$, we extend it to an open cover $\tilde{\mathcal{O}}$ of X via $\tilde{\mathcal{O}} = \mathcal{O} \cup \{\, X \setminus \mathcal{C} \,\}$. Using the paracompactness of (X, τ) we get a locally finite open refinement $\tilde{\mathcal{X}}$ that covers X. We restrict these sets to $\mathcal{C}$ to obtain a locally finite open refinement $\mathcal{X}$ of $\mathcal{O}$ that covers $\mathcal{C}$. □

Theorem 1.6. *If (X, τ) is a topological space, and if $\mathcal{A} \subseteq \mathcal{P}(X)$ is locally finite, then the set:*

$$\mathcal{A}' = \{\, Cl_\tau(A) \mid A \in \mathcal{A}\,\} \tag{7}$$

is locally finite as well

Proof. Let $x \in X$. Since $\mathcal{A}$ is locally finite, there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{U} \cap A = \emptyset$ for all but finitely many elements of $\mathcal{A}$. Suppose $A \in \mathcal{A}$ and $\mathcal{U} \cap A = \emptyset$. Let us show that $\mathcal{U} \cap \mathrm{Cl}_\tau(A) = \emptyset$. Since $\mathcal{U}$ is open, and since $\mathcal{U} \cap A = \emptyset$, we have that $X \setminus \mathcal{U}$ is closed and $A \subseteq X \setminus \mathcal{U}$. But since $X \setminus \mathcal{U}$ is closed we have $\mathrm{Cl}_\tau(A) \subseteq X \setminus \mathcal{U}$. But then $\mathcal{U} \cap \mathrm{Cl}_\tau(A) = \emptyset$. This means for all $A \in \mathcal{A}$, $A \cap \mathcal{U}$ is non-empty if and only if $\mathrm{Cl}_\tau(A) \cap \mathcal{U}$ is non-empty. Since only finitely many elements of $\mathcal{A}$ have non-empty intersection with $\mathcal{U}$, the exact same number of sets in $\mathcal{A}'$ will have non-empty intersection with $\mathcal{U}$. Hence the collection $\mathcal{A}'$ is locally finite. $\square$

Theorem 1.7. *If (X, τ) is a topological space, and if $\mathcal{A} \subseteq \mathcal{P}(X)$ is locally finite, then:*

$$Cl_\tau\Big(\bigcup_{A \in \mathcal{A}} A\Big) = \bigcup_{A \in \mathcal{A}} Cl_\tau(A) \tag{8}$$

Proof. Even without the locally finite assumption, we may prove that:

$$\bigcup_{A \in \mathcal{A}} \mathrm{Cl}_\tau(A) \subseteq \mathrm{Cl}_\tau\Big(\bigcup_{A \in \mathcal{A}} A\Big) \tag{9}$$

Since $A \in \mathcal{A}$ we have that $A \subseteq \bigcup \mathcal{A}$. But then $\mathrm{Cl}_\tau(A) \subseteq \mathrm{Cl}_\tau(\bigcup \mathcal{A})$. Since this is true of all $A \in \mathcal{A}$, the union on the left-hand side of the equation must be a subset of the right-hand side. To prove the reverse inclusion requires the locally finite condition. Let $x \in \mathrm{Cl}_\tau(\bigcup \mathcal{A})$. Since $\mathcal{A}$ is locally finite, there is an open set $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{U} \cap A = \emptyset$ for all but finitely many $A \in \mathcal{A}$. But $\mathcal{U}$ is open, $x \in \mathcal{U}$, and $x \in \mathrm{Cl}_\tau(\bigcup \mathcal{A})$ meaning $\mathcal{U} \cap \bigcup \mathcal{A}$ is non-empty. But since only finitely many elements of $\mathcal{A}$ can intersect $\mathcal{U}$, and since the number of such elements is not zero by the previous statement, we may label the sets $A_0, \dots, A_n$ for some $n \in \mathbb{N}$. But then:

$$\mathrm{Cl}_\tau\Big(\bigcap_{A \in \mathcal{A}} (A \cap \mathcal{U})\Big) = \mathrm{Cl}_\tau\Big(\bigcap_{k=0}^{n} (A_k \cap \mathcal{U})\Big) = \bigcup_{k=0}^{n} \mathrm{Cl}_\tau(A_k \cap \mathcal{U}_n) \tag{10}$$

But also:

$$\bigcup_{k=0}^{n} \mathrm{Cl}_\tau(A_k \cap \mathcal{U}_n) \subseteq \bigcup_{k=0}^{n} \mathrm{Cl}_\tau(A_k) \subseteq \bigcup_{A \in \mathcal{A}} \mathrm{Cl}_\tau(A) \tag{11}$$

From this we may conclude that $x \in \bigcup_{A \in \mathcal{A}} \mathrm{Cl}_\tau(A)$, and hence we have equality. $\square$

Theorem 1.8. *If (X, τ) is paracompact and Hausdorff, then it is regular.*

Proof. For let $x \in X$, $\mathcal{C} \subseteq X$ be closed, and $x \notin \mathcal{C}$. Since (X, τ) is Hausdorff, for all $y \in \mathcal{C}$ there are $\mathcal{U}_y, \mathcal{V}_y \in \tau$ such that $x \in \mathcal{U}_y$, $y \in \mathcal{V}_y$, and $\mathcal{U}_y \cap \mathcal{V}_y = \emptyset$. But then:

$$\mathcal{O} = \{\, \mathcal{V}_y \mid y \in \mathcal{C} \,\} \tag{12}$$

is an open cover of $\mathcal{C}$. But (X, τ) is paracompact and $\mathcal{C}$ is closed, so there is a locally finite open refinement $\mathcal{X}$ that covers $\mathcal{C}$. But by the definition of $\mathcal{O}$, since $\mathcal{X}$ is a refinement of $\mathcal{O}$, for all $\mathcal{W} \in \mathcal{X}$ there is a $\mathcal{V}_y \in \mathcal{O}$ such that $\mathcal{W} \subseteq \mathcal{V}_y$. Hence all elements of $\mathcal{X}$ are subsets of $\mathcal{V}_y$ for some $y \in \mathcal{C}$. But then, since $x \in \mathcal{U}_y$ and $\mathcal{U}_y \cap \mathcal{V}_y = \emptyset$, we have $x \notin \mathrm{Cl}_\tau(\mathcal{V}_y)$. But $\mathcal{X}$ is locally finite, hence:

$$\mathrm{Cl}_\tau\Big(\bigcup_{A \in \mathcal{X}} A\Big) = \bigcup_{A \in \mathcal{X}} \mathrm{Cl}_\tau(A) \tag{13}$$

And hence $x \notin \mathrm{Cl}_\tau\big(\bigcup_{A \in \mathcal{X}} A\big)$. Let $\mathcal{U} = X \setminus \mathrm{Cl}_\tau\Big(\bigcup \mathcal{X}\Big)$ and $\mathcal{V} = \bigcup \mathcal{X}$. Then $\mathcal{U}$ and $\mathcal{V}$ are open and disjoint, $x \in \mathcal{U}$, and $\mathcal{C} \subseteq \mathcal{V}$. Hence, (X, τ) is regular. $\square$

Theorem 1.9 (**Dieudonne's Theorem**). *If (X, τ) is paracompact and Hausdorff, then it is normal.*

Proof. We apply the same idea as before. Since (X, τ) is paracompact and Hausdorff, it is regular. Given two closed disjoint sets $\mathcal{C}, \mathcal{D} \subseteq X$ for all $x \in \mathcal{C}$ we find $\mathcal{U}_x, \mathcal{V}_x \in \tau$ such that $x \in \mathcal{U}_x$, $\mathcal{D} \subseteq \mathcal{V}_x$, and $\mathcal{U}_x \cap \mathcal{V}_x = \emptyset$. We use paracompactness and apply a similar argument to the previous theorem to prove normality. $\square$

We now take the steps towards proving the two main metrization theorems. The Nagata-Smirnov theorem, and the Smirnov theorem. The Smirnov theorem uses paracompactness to characterize metrizable spaces, the Nagata-Smirnov theorem uses a very similar idea as the Urysohn metrization theorem. Urysohn's theorem said a regular Hausdorff space that is second countable is metrizable. All metrizable spaces are regular and Hausdorff, so this can not be omitted, however the second countability can be weakened. The idea that is required for metrizability is σ locally finite bases. Related to this concept is the notion of a σ locally finite open cover.

Definition 1.5 (σ **Locally Finite Open Cover**) A σ locally finite open cover of a topological space (X, τ) is an open cover $\mathcal{O} \subseteq \tau$ such that there exists countably many locally finite collections $\Delta_n \subseteq \tau$ such that $\mathcal{O} = \bigcup_{n \in \mathbb{N}} \Delta_n$. $\blacksquare$

Now for some results related to this notion, paracompactness, and metrization. First, a little lemma.

Theorem 1.10. *If (X, τ) is a metrizable topological space, if d is a metric that induces the topology, if $A \subseteq X$, and if $r \in \mathbb{R}^+$, then the set:*

$$S_r(A) = \{\, x \in X \mid B_r^{(X,\,d)}(x) \subseteq A \,\} \tag{14}$$

is closed.

Proof. For if not then there is a sequence $a : \mathbb{N} \to S_r(A)$ that converges to a point $x \in X$ but $x \notin S_r(A)$. But if $x \notin S_r(A)$, then there is a point $y \in X$ such that $d(x, y) < r$ and $y \notin A$. Let $\varepsilon = r - d(x, y)$. Since $\varepsilon > 0$ and $a_n \to x$, there is an $N \in \mathbb{N}$ such that $n \in \mathbb{N}$ and $n > N$ implies $d(x, a_n) < \varepsilon$. Let $n = N + 1$. Then $n > N$ and hence $d(x, a_n) < \varepsilon$. But then:

$$\begin{aligned} d(a_n, y) &\leq d(a_n, x) + d(x, y) && (15)\\ &< \varepsilon + d(x, y) && (16)\\ &= (r - d(x, y)) + d(x, y) && (17)\\ &= r && (18) \end{aligned}$$

And hence $d(a_n, y) < r$. But then $y \in B_r^{(X,\,d)}(a_n)$. But $a_n \in S_r(A)$, so $B_r^{(X,\,d)}(a_n) \subseteq A$. But then $y \in A$, a contradiction. Hence, $S_r(A)$ is closed. □

Theorem 1.11. *If (X, τ) is metrizable, and if $\mathcal{O} \subseteq \tau$ is an open cover, then there is a σ locally finite open cover $\mathcal{X}$ that is a refinement of $\mathcal{O}$.*

Proof. This theorem is part of the proof that metrizable spaces are paracompact (which is Stone's paracompactness theorem). The original proof is very long but can be shortened by using the well ordering theorem. This makes the theorem less-than-intuitive, but pages and pages shorter. The curious reader should consult A. H. Stone's original papers for the more straight-forward but longer and more involved proof.

Since $\mathcal{O}$ is a set, there is a well-order on it $\prec$. Since (X, τ) is metrizable there is a metric d that induces the topology τ. For all $n \in \mathbb{N}$ and for all $\mathcal{U} \in \mathcal{O}$ define:

$$S_n(\mathcal{U}) = \{\, x \in X \mid B_{\frac{1}{n+1}}^{(X,\,d)}(x) \subseteq \mathcal{U} \,\} \tag{19}$$

That is, the set of all points in $\mathcal{U}$ that can be surrounded with a ball of radius $1/(n+1)$ that is completely contained in $\mathcal{U}$. Each $S_n(\mathcal{U})$ is closed by the previous theorem. Define $T_n(\mathcal{U})$ via:

$$T_n(\mathcal{U}) = S_n(\mathcal{U}) \setminus \bigcup \{\, \mathcal{V} \in \mathcal{O} \mid \mathcal{V} \prec \mathcal{U} \,\} \tag{20}$$

We have used the well-order. We take the union of all sets that are *less than* $\mathcal{U}$ with respect to the well-order $\prec$ in the last part of this equation. For all $x \in T_n(\mathcal{U})$ and $y \in T_n(\mathcal{V})$, with $\mathcal{U}, \mathcal{V} \in \mathcal{O}$ distinct, we have from this construction that $d(x, y) \geq \frac{1}{n+1}$. The set $T_n(\mathcal{U})$ is the difference of an open set from a closed set, which is therefore closed. We want open sets. Define $E_n(\mathcal{U})$ via:

$$E_n(\mathcal{U}) = \bigcup_{x \in T_n(\mathcal{U})} B_{\frac{1}{3n+3}}^{(X,\,\tau)}(x) \tag{21}$$

Then $E_n(\mathcal{U})$ is open, being the union of open balls. By choosing the radii to be $\frac{1}{3n+3}$, we ensure that $E_n(\mathcal{U})$ and $E_n(\mathcal{V})$ are disjoint whenever $\mathcal{U}, \mathcal{V} \in \tau$ are distinct. We use this to obtain our σ locally finite cover that is a refinement of $\mathcal{O}$. First, define:

$$\mathcal{A}_n = \{\, E_n(\mathcal{U}) \mid \mathcal{U} \in \mathcal{O} \,\} \tag{22}$$

And define $\mathcal{X} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. From the construction, given $\mathcal{U} \in \mathcal{O}$, we have $E_n(\mathcal{U}) \subseteq \mathcal{U}$, and hence $\mathcal{X}$ is a refinement of $\mathcal{O}$. Each $\mathcal{A}_n$ is locally finite, the ball of radius $\frac{1}{6n+6}$ centered at x intersects only one element of $\mathcal{A}_n$. Lastly, $\mathcal{X}$ covers X. Given $x \in X$, since $\mathcal{O}$ covers X, there is a $\mathcal{U} \in \mathcal{O}$ such that $x \in \mathcal{U}$. But d induces τ, so there is an $r > 0$ such that the r ball centered at x is contained in $\mathcal{U}$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N+1} < r$. Then $x \in E_N(\mathcal{U})$, showing that there is an element of $\mathcal{X}$ that contains x. So $\mathcal{X}$ is a σ locally finite open cover that is a refinement of $\mathcal{X}$. $\square$

This idea comes from Mary Ellen Rudin, one of the great topologists of the second half of the 20^{th} century. Stone's paracompactness theorem now comes in two more steps. These theorems are a bit of work, but we get two pretty results out of them. First, Stone's paracompactness theorem, but also the fact that every regular Lindelöf space is paracompact, essentially for free.

Theorem 1.12. *If (X, τ) is a topological space, if $\mathcal{O} \subseteq \tau$ is an open cover, and if $\mathcal{X}$ is a σ locally finite open cover that is a refinement of $\mathcal{O}$, then there is a locally finite cover (but not necessarily an open one) Δ that is a refinement of $\mathcal{O}$.*

Proof. Since $\mathcal{X}$ is σ locally finite and an open cover there are countably many $\mathcal{A}_n$, each of which is locally finite, such that $\mathcal{X} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. Let $\mathcal{U}_n \in \tau$ be defined by:

$$\mathcal{U}_n = \bigcup \mathcal{A}_n \tag{23}$$

Since the elements of $\mathcal{A}_n$ are open subsets, $\mathcal{U}_n$ is open. For all $\mathcal{V} \in \mathcal{A}_n$, define:

$$S_n(\mathcal{V}) = \mathcal{V} \setminus \bigcup_{k=0}^{n-1} \mathcal{U}_k \tag{24}$$

$S_n(\mathcal{V})$ is the set difference of an open set from an open set, so it may be open, closed, both, or neither. We simply don't know. Define Δ_n to be:

$$\Delta_n = \{\, S_n(\mathcal{V}) \mid \mathcal{V} \in \mathcal{A}_n \,\} \tag{25}$$

Let $\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$. We must now show Δ is locally finite and covers X. First, Δ is a cover of X. Given $x \in X$, since the sets $\mathcal{U}_n$ cover X, there is some $n \in \mathbb{N}$ such that $x \in \mathcal{U}_n$. By the well-ordering property of the natural numbers there is a least such number $N \in \mathbb{N}$ such that $x \in \mathcal{U}_N$. Then for all $n < N$ we have $x \notin \mathcal{U}_n$, and hence by definition $x \in S_N(\mathcal{V})$ where $\mathcal{V} \in \mathcal{A}_N$ is such that $x \in \mathcal{V}$. So Δ covers X. Next, it is locally finite. Given $x \in X$, since each $\mathcal{A}_n$ is locally finite we have that there is a $\mathcal{V}_n \in \tau$ such that $x \in \mathcal{V}_n$ and $\mathcal{V}_n$ intersects only finitely many elements of $\mathcal{A}_n$. Again, let $N \in \mathbb{N}$ be the least integer such that $x \in \mathcal{U}_N$. Then for all $n > N$ we have $\mathcal{U}_N$ has empty intersection with the elements of Δ_n, by definition of S_n. So the set:

$$\tilde{\mathcal{U}} = \mathcal{U}_N \cap \bigcap_{k=0}^{N} \mathcal{V}_k \tag{26}$$

is an open set that contains x and is such that only finitely many elements of Δ have non-empty intersection with $\tilde{\mathcal{U}}$. Hence Δ is locally finite. □

This would be much stronger if the set Δ consists of open sets. In this construction it almost certainly does not. The difference of an open set from an open set is rarely open in usual spaces. For example, in $\mathbb{R}$, given $a < c < b < d$, $(a,\, b) \setminus (c,\, d) = (a,\, c]$, which is neither closed nor open. We need to modify this idea.

Theorem 1.13. *If $(X,\, \tau)$ is a regular topological space such that for all open covers $\mathcal{O}$ of X there is a locally finite refinement Δ of $\mathcal{O}$ that covers X, then every open cover $\mathcal{O}$ has a locally finite open refinement $\mathcal{X}$ that covers X.*

Proof. Let $\mathcal{O}$ be an open cover and let Δ be a locally finite refinement (it does not need to be an open one). Since Δ is locally finite, for all $x \in X$ there is a $\mathcal{U}_x \in \tau$ such that $\mathcal{U}_x$ has non-empty intersection with only finitely many elements of Δ. The set:

$$\mathcal{A} = \{\, \mathcal{U}_x \mid x \in X \,\} \tag{27}$$

is thus an open cover of X. Let Δ' be a locally finite refinement of $\mathcal{A}$, which exists by hypothesis. Define

$$\Delta'' = \{\, \mathrm{Cl}_\tau(A) \mid A \in \Delta' \,\} \tag{28}$$

Since Δ' is locally finite, so is Δ'' and Δ'' consists of closed sets. For all $A \in \Delta$, define $\mathcal{B}_A$ via:

$$\mathcal{B}_A = \{\, B \in \Delta'' \mid B \subseteq X \setminus A \,\} \tag{29}$$

and define $\mathcal{V}_A$ via:

$$\mathcal{V}_A = X \setminus \bigcup \mathcal{B}_A \tag{30}$$

Since Δ'' is locally finite, and since the elements of it are closed, we have $\bigcup \mathcal{B}_A$ is closed, and hence $\mathcal{V}_A$ is open. Since each element of $\mathcal{B}_A$ is disjoint from A, we have that $A \subseteq \mathcal{V}_A$. Since Δ is a refinement of $\mathcal{O}$, for all $A \in \Delta$ there is a $\mathcal{W}_A \in \mathcal{O}$ such that $A \subseteq \mathcal{W}_A$. Define:

$$\mathcal{X} = \{\, \mathcal{V}_A \cap \mathcal{W}_A \mid A \in \Delta \,\} \tag{31}$$

Then $\mathcal{X}$ is a refinement of $\mathcal{O}$ and also an open cover of X, by definition of $\mathcal{V}_A$ and $\mathcal{W}_A$. We must show it is locally finite. For given $x \in X$, since Δ'' is locally finite, there is a $\mathcal{W} \in \tau$ such that $x \in \mathcal{W}$ and only finitely many elements of Δ'' have non-empty intersection with $\mathcal{W}$. But Δ'' covers X, so the elements of Δ'' with non-empty intersection with $\mathcal{W}$ must also cover $\mathcal{W}$. Because of this we need only show that any given element of Δ'' intersects only finitely many elements of $\mathcal{X}$. Let let $B \in \Delta''$. If B intersects some element of $\mathcal{X}$, $\mathcal{V}_A \cap \mathcal{W}_A$ for some $A \in \Delta$, then by definition of $\mathcal{V}_A$ we have $B \cap (X \setminus A) = \emptyset$. But then $B \cap A \neq \emptyset$. But the elements of Δ'' intersect only finitely many elements of Δ, B can only intersect finitely many elements of $\mathcal{X}$. Hence, $\mathcal{X}$ is a locally finite open refinement of $\mathcal{O}$ that covers X. □

Theorem 1.14. *If (X, τ) is regular and Lindelöf, then it is paracompact.*

Proof. Given a cover $\mathcal{O}$, since (X, τ) is Lindelöf there is a countable subcover Δ. Since this is countable, it is also a σ locally finite open cover which is a refinement of $\mathcal{O}$, since it is a subset of it. It is σ locally finite since we can find a surjection $\mathcal{U} : \mathbb{N} \to \Delta$ and letting $\Delta_n = \{\mathcal{U}_n\}$, we have $\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n$. Each Δ_n is locally finite since it is, well, finite. By a previous theorem, since (X, τ) is regular, there is therefore a locally finite open refinement of Δ that covers X. So (X, τ) is paracompact. □

Theorem 1.15 (Stone's Paracompactness Theorem). *If $(X \tau)$ is metrizable, then it is paracompact.*

Proof. Since (X, τ) is metrizable, every open cover $\mathcal{O}$ has a σ locally finite open cover $\mathcal{X}$ that is a refinement of $\mathcal{O}$. By a previous theorem, since metrizable spaces are regular, there is a locally finite open refinement of this that is an open cover, and hence (X, τ) is paracompact. □