Point-Set Topology: Lecture 25

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1 Partitions of Unity

We're a stone's throw away (no pun intended A. H. Stone) from the metrization theorems. It seems historically the ideas of paracompactness and local finiteness grew out of the study of metrization theorems, but these notions have found enormous use elsewhere. Indeed, the use of paracompactness outside of general topology is probably more well known than the metrization theorems. In manifold theory and geometry the importance comes from the relation of paracompactness to *partitions of unity*. Some of the key theorems of differential topology (like the Whitney embedding theorem) and Riemannian geometry (every smooth manifold is a Riemannian manifold) rely on the fact that topological manifolds have partitions of unity. This is a corollary of the fact that topological manifolds are paracompact and Hausdorff. In this section we introduce the idea and prove some basic theorems.

First, some notation. Given two sets A and B we can prove there exists a set $\mathcal{F}(A,\,B)$ that is the *set of all functions* from A to B. Hence if we have two topological spaces $(X,\,\tau_X)$ and $(Y,\,\tau_Y)$ we can collect all *continuous* functions from X to Y. This is denoted $C^0(X,\,Y)$, or sometimes just $C(X,\,Y)$. The reason for the 0 is that in some cases (i.e. smooth manifolds and topological vector spaces) it is possible to say a function from one topological space to another is *differentiable* or *twice differentiable* or even *smooth.* We would then use the notations $C^1(X,\,Y)$, $C^2(X,\,Y)$, and $C^\infty(X,\,Y)$, respectively. For general topological spaces there is no notion of derivative, so writing $C(X,\,Y)$ is fine.

A word of warning. Some authors use $C(X)$ to denote continuous functions from $(X,\,\tau)$ to $\mathbb{R}$, where $\mathbb{R}$ has the standard Euclidean topology. Some authors use $C(X)$ to denote continuous functions from $(X,\,\tau)$ to $\mathbb{C}$, where $\mathbb{C}=\mathbb{R}^2$ has the standard topology of the Euclidean plane. These notions are not equivalent, and it becomes the responsibility of the reader to remember which notation the author is using. I won't be doing this. When I want to speak of continuous function into $\mathbb{R}$, I'll write $C(X,\,\mathbb{R})$. If I want to consider continuous functions into the complex plane, it'll be denoted $C(X,\,\mathbb{C})$.

Definition 1.1 (**Support of a Function**) The support of a function $f:X\rightarrow$

$\mathbb{R}$ from a topological space $(X,\,\tau)$ to the real line is the set $\text{supp}_\tau(f) \subseteq X$ defined by:

$$\text{supp}_\tau(f) = \text{Cl}_\tau\Big(f^{-1}\big[\mathbb{R}\setminus\{\,0\,\}\big]\Big) \tag{1}$$

That is, the closure of the set of all points in X that don't map to zero. ■

For those familiar with algebra, we can replace $\mathbb{R}$ with any ring (or, more commonly, any field). In analysis one often uses $\mathbb{C}$ instead.

Definition 1.2 (**Partition of Unity**) A partition of unity in a topological space $(X,\,\tau)$ is a subset $\mathcal{R} \subseteq C(X,\,\mathbb{R})$ of continuous functions from X to $\mathbb{R}$ such that:

1. For all $f \in X$ and for all $x \in X$ it is true that $f(x) \geq 0$.
2. The set $\mathcal{O} \subseteq \mathcal{P}(X)$ defined by:

   $$\mathcal{O} = \{\,\text{supp}_\tau(f) \mid f \in \mathcal{R}\,\} \tag{2}$$

   is locally finite.
3. For all $x \in X$ we have:

   $$\sum_{f\in\mathcal{R}} f(x) = 1 \tag{3}$$

■

This last part may be confusing since $\mathcal{R}$ can be uncountably big and we've no notion of summing over sets that big (It is possible to do this, however. For those who have studied measure theory you may recall that if the sum of uncountably many non-negative real numbers converges, then all but countably many are zero). There is no issue of convergence because of the second property. Since the supports are locally finite, given any $x \in X$ there are only finitely many $f \in \mathcal{R}$ such that $f(x) \neq 0$. So the equation in part 3 is a finite sum for each point.

It is useful to attach partitions of unity to open covers. In almost all applications we consider partitions of unity that are *subordinate* to the cover.

Definition 1.3 (**Partition of Unity Subordinate to an Open Cover**) A partition of unity that is subordinate to an open cover $\mathcal{O}$ in a topological space $(X,\,\tau)$ is a partition of unity $\mathcal{R}$ such that for all $f \in \mathcal{R}$ there is a $\mathcal{U} \in \mathcal{O}$ such that $\text{supp}_\tau(f) \subseteq \mathcal{U}$. ■

Every paracompact Hausdorff space has a subordinate partition of unity for every open cover. Conversely, every Hausdorff space that always admits subordinate partitions of unity is paracompact. We will prove these two facts in todays notes. The *shrinking lemma* is required, which says that paracompact Hausdorff spaces are precisely paracompact.

Definition 1.4 (**Precise Refinement**) A precise refinement of a set $\mathcal{A} \subseteq \mathcal{P}(X)$ in a topological space $(X,\,\tau)$ is a set $\mathcal{A}' \subseteq \mathcal{P}(X)$ such that for all $A' \in \mathcal{A}'$ there is an $A \in \mathcal{A}$ such that $\mathrm{Cl}_\tau(A') \subseteq A$. ■

With refinements we only required $A' \subseteq A$. This allows for the possibility that $A = A'$. With precise refinements, unless the sets under consideration happen to be closed, to make $\mathrm{Cl}_\tau(A') \subseteq A$ requires *shrinking* A a little.

Definition 1.5 (**Precisely Paracompact Topological Space**) A precisely paracompact topological space is a topological space $(X,\,\tau)$ such that for all open covers $\mathcal{O} \subseteq \tau$ of X there exists a precise locally finite open refinement $\mathcal{X}$ of $\mathcal{O}$ that covers X. ■

Theorem 1.1 (**The Shrinking Lemma**). *If $(X,\,\tau)$ is paracompact and Hausdorff, then it is precisely paracompact.*

Proof. For if not then there is an open cover $\mathcal{O} \subseteq \tau$ with no precise locally finite open refinement that covers X. But $(X,\,\tau)$ is paracompact so there is a locally finite open refinement $\mathcal{X}$ of $\mathcal{O}$. But $(X,\,\tau)$ is Hausdorff and paracompact, so it is regular. Hence for all $x \in X$ and for all $\mathcal{U} \in \tau$ with $x \in \mathcal{U}$ there is a $\mathcal{V} \in \tau$ such that $x \in \mathcal{V}$ and $\mathrm{Cl}_\tau(\mathcal{V}) \subseteq \mathcal{U}$. Using this, since $\mathcal{X}$ is an open cover of X, for all $x \in X$ there is a $\mathcal{U}_x \in \mathcal{X}$ such that $x \in \mathcal{U}_x$. Let $\mathcal{V}_x \in \tau$ be such that $x \in \mathcal{V}_x$ and $\mathrm{Cl}_\tau(\mathcal{V}_x) \subseteq \mathcal{U}_x$. Define $\tilde{\mathcal{O}}$ via:

$$\tilde{\mathcal{O}} = \{\, \mathcal{V}_x \mid x \in X \,\} \tag{4}$$

This is now a precise open refinement of $\mathcal{O}$, but it is probably not locally finite (we've picked an open set for *every* point in the space). But $\tilde{\mathcal{O}}$ is an open cover of X since given $x \in X$ it is contained in $\mathcal{V}_x$ which is an element of $\tilde{\mathcal{O}}$. Since $(X,\,\tau)$ is paracompact there is a locally finite open refinement $\tilde{\mathcal{X}}$ of $\tilde{\mathcal{O}}$ that is an open cover of X. But then for all $\mathcal{U} \in \tilde{\mathcal{X}}$ there is a $\mathcal{V} \in \tilde{\mathcal{O}}$ such that $\mathcal{U} \subseteq \mathcal{V}$. But by definition of $\tilde{\mathcal{O}}$ there is a $\mathcal{W} \in \mathcal{X}$ such that $\mathrm{Cl}_\tau(\mathcal{V}) \subseteq \mathcal{W}$. But $\mathcal{X}$ is refinement of $\mathcal{O}$, so there is a $\mathcal{E} \in \mathcal{O}$ such that $\mathcal{W} \subseteq \mathcal{E}$. But then:

$$\mathrm{Cl}_\tau(\mathcal{V}) \subseteq \mathrm{Cl}_\tau(\mathcal{U}) \subseteq \mathcal{W} \subseteq \mathcal{E} \tag{5}$$

and hence $\mathrm{Cl}_\tau(\mathcal{V}) \subseteq \mathcal{E}$. But then $\tilde{\mathcal{X}}$ is a precise locally finite open refinement of $\mathcal{O}$ that covers X, a contradiction. Hence, $(X,\,\tau)$ is precisely paracompact. □

Theorem 1.2. *If $(X,\,\tau)$ is a Hausdorff topological space, then it is paracompact if and only if every open cover has a subordinate partition of unity.*

Proof. Suppose $(X,\,\tau)$ is Hausdorff and every open cover has a subordinate partition of unity. Suppose it is not paracompact, meaning there is an open cover $\mathcal{O} \subseteq \tau$ with no locally finite open refinement that covers X. But since $\mathcal{O}$ is an open cover, by hypothesis there is a subordinate partition of unity $\mathcal{R} \subseteq C(X,\,\mathbb{R})$. Define $\mathcal{X}$ via:

$$\mathcal{X} = \big\{\, f^{-1}\big[\mathbb{R} \setminus \{\,0\,\}\big] \mid f \in \mathcal{R} \,\big\} \tag{6}$$

Since each element of $\mathcal{R}$ is continuous, and since $\mathbb{R} \setminus \{ 0 \}$ is open, the elements of $\mathcal{X}$ are open. $\mathcal{X}$ also covers X since given $x \in X$ we have:

$$\sum_{f \in \mathcal{R}} f(x) = 1 \tag{7}$$

So there must be some $f \in \mathcal{R}$ such that $f(x) \neq 0$. But then $x \in f^{-1}\big[\mathbb{R} \setminus \{ 0 \}\big]$, meaning $\mathcal{X}$ is an open cover. It is a refinement of $\mathcal{O}$ since given $\mathcal{U} \in \mathcal{X}$ we have that there is some $f \in \mathcal{R}$ such that $\mathcal{U} = f^{-1}\big[\mathbb{R} \setminus \{ 0 \}\big]$. But since $\mathcal{R}$ is subordinate to $\mathcal{O}$ there is a $\mathcal{V} \in \mathcal{O}$ such that $\mathrm{supp}_\tau(f) \subseteq \mathcal{V}$. But then:

$$\mathcal{U} = f^{-1}\big[\mathbb{R} \setminus \{ 0 \}\big] \subseteq \mathrm{Cl}_\tau\Big(f^{-1}\big[\mathbb{R} \setminus \{ 0 \}\big]\Big) = \mathrm{supp}_\tau(f) \subseteq \mathcal{V} \tag{8}$$

and hence $\mathcal{U} \subseteq \mathcal{V}$. Lastly, $\mathcal{X}$ is locally finite. For let $x \in X$. Since $\mathcal{R}$ is a partition of unity there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and only finitely many $f \in \mathcal{R}$ are such that $\mathrm{supp}_\tau(f) \cap \mathcal{U}$ is non-empty. But $f^{-1}\big[\mathbb{R} \setminus \{ 0 \}\big] \subseteq \mathrm{supp}_\tau(f)$, meaning only finitely many elements of $\mathcal{X}$ are such that $f^{-1}\big[\mathbb{R} \setminus \{ 0 \}\big]$ intersects $\mathcal{U}$, and hence $\mathcal{X}$ is locally finite. So $\mathcal{X}$ is a locally finite open refinement of $\mathcal{O}$ that covers X, a contradiction. So $(X,\, \tau)$ is paracompact. Now, suppose $(X,\, \tau)$ is paracompact and Hausdorff. Let $\mathcal{O} \subseteq \tau$ be an open cover of X. Since $(X,\, \tau)$ is paracompact and Hausdorff, it is precisely paracompact, so there is a precise locally finite open refinement $\mathcal{X}_0$ of $\mathcal{O}$ that covers X. But then $\mathcal{X}_0$ is an open cover of X, and since $(X,\, \tau)$ is precisely paracompact there is a precise locally finite open refinement $\mathcal{X}_1$ of $\mathcal{X}_0$. Then for all $\mathcal{U}_1 \in \mathcal{X}_1$ there is a $\mathcal{U}_0 \in \mathcal{X}_0$ and a $\mathcal{U} \in \mathcal{O}$ such that $\mathrm{Cl}_\tau(\mathcal{U}_1) \subseteq \mathcal{U}_0$ and $\mathrm{Cl}_\tau(\mathcal{U}_0) \subseteq \mathcal{U}$ (See Fig. 1). But $\mathrm{Cl}_\tau(\mathcal{U}_1)$ is closed, and since $\mathcal{U}_0$ is open, $X \setminus \mathcal{U}_0$ is closed. But $\mathrm{Cl}_\tau(\mathcal{U}_1) \subseteq \mathcal{U}_0$ so $X \setminus \mathcal{U}_0$ and $\mathrm{Cl}_\tau(\mathcal{U}_1)$ are disjoint. But since $(X,\, \tau)$ is paracompact and Hausdorff, by Dieudonné's theorem it is normal. But then by Urysohn's lemma, since $\mathrm{Cl}_\tau(\mathcal{U}_1)$ and $X \setminus \mathcal{U}_0$ are closed and disjoint, there is a continuous function $f_{\mathcal{U}_1} : X \to [0,\, 1]$ such that $f_{\mathcal{U}_1}\big[\mathrm{Cl}_\tau(\mathcal{U}_1)\big] = \{ 1 \}$ and $f_{\mathcal{U}_1}\big[X \setminus \mathcal{U}_0\big] = \{ 0 \}$. That is, $f_{\mathcal{U}_1}$ is continuous, evaluates to 1 on the closure of $\mathcal{U}_1$, zero outside of $\mathcal{U}_0$, and some values between 0 and 1 for points in $\mathcal{U}_0 \setminus \mathrm{Cl}_\tau(\mathcal{U}_1)$. The set:

$$\tilde{\mathcal{R}} = \{ f_{\mathcal{U}_1} \mid \mathcal{U}_1 \in \mathcal{X}_1 \} \tag{9}$$

is almost a partition of unity. Since Urysohn's lemma restricts the range of the functions to $[0,\, 1]$, we have $f(x) \geq 0$ for all $x \in X$ and $f \in \tilde{\mathcal{R}}$. The set of supports are locally finite, since the support of $f_{\mathcal{U}_1}$ is $\mathrm{Cl}_\tau(\mathcal{U}_0)$, and these form a locally finite set. The last thing we need is that the sums at all points evaluate to one. $\tilde{\mathcal{R}}$ almost certainly does not have this property. Given $x \in X$, since $\mathcal{X}_1$ is an open cover of X, there is a $\mathcal{U}_1 \in \mathcal{X}_1$ such that $x \in \mathcal{U}_1$. But then $f_{\mathcal{U}_1}(x) = 1$. Hence the function $\Phi : X \to \mathbb{R}^+$ defined by:

$$\Phi(x) = \sum_{f \in \tilde{\mathcal{R}}} f(x) \tag{10}$$

is positive for all $x \in X$. It is also continuous since locally it is the finite sum

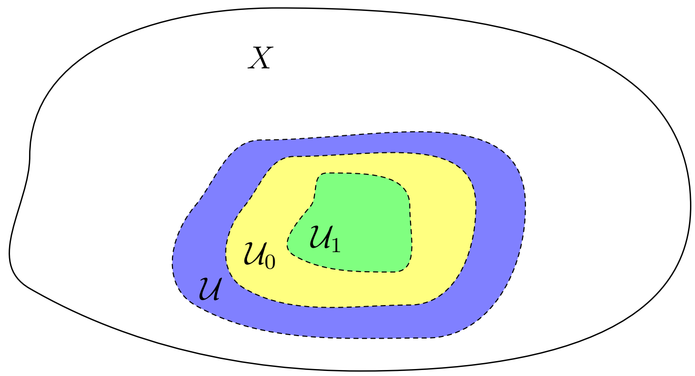

Figure 1: Nested Precise Refinements for Thm. 1.

of continuous functions. Since $\Phi(x)$ is never zero we can define:

$$g_{\mathcal{U}_1}(x) = \frac{f_{\mathcal{U}_1}(x)}{\Phi(x)} \tag{11}$$

The set:

$$\mathcal{R} = \{\, g_{\mathcal{U}_1} \mid \mathcal{U}_1 \in \mathcal{X}_1 \,\} \tag{12}$$

is then a partition of unity subordinate to $\mathcal{O}$, contradicting the claim that none such partition of unity exists. Hence, if $(X,\, \tau)$ is a paracompact Hausdorff space, then every open cover has a subordinate partition of unity. □

That's about all we'll say about partitions of unity, as far as point-set topology is concerned. In the world of differential topology and geometry there are two important theorems that are worth noting. A topological manifold is a topological space $(X,\, \tau)$ that is Hausdorff, second countable, and locally Euclidean. This last property means for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U},\, \tau_{\mathcal{U}})$ is, as a subspace, homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. If n is fixed (if $(X,\, \tau)$ is connected it has to be fixed) it is called the *dimension* of the manifold.

A smooth manifold is a topological manifold with a *smooth structure* which makes it possible to talk about the derivatives of functions, tangent vector, and vector fields, topics common in a multivariable calculus course. A Riemannian manifold is a smooth manifold with a *Riemannian metric* which makes it possible to measure angles of tangent vectors and compute the lengths of curves. Notions like geodesics and parallel transport occur in Riemannian geometry. So every Riemannian manifold is a smooth manifold and every smooth manifold is a topological manifold. Topological manifolds are paracompact and Hausdorff, so every open cover has a subordinate partition of unity. For smooth manifolds one can improve this. Every open cover in a smooth manifold $(X,\, \tau)$ has a subordinate partition of unity $\mathcal{R}$ where the functions $f \in \mathcal{R}$ can be taken to be *smooth* (which now makes sense since in smooth manifolds we can take derivatives). Using this it can be shown (without much work, surprisingly) that every smooth manifold can be made into a Riemannian manifold. It can also be shown that every smooth manifold of dimension $n \in \mathbb{N}$ is actually just a topological subspace of $\mathbb{R}^{2n+1}$ (one can improve this to $\mathbb{R}^{2n}$, but this is harder).

With this, a natural question is whether or not every topological manifold can be made into a smooth manifold. Shockingly, the answer is no. There are topological manifolds that are so rough and rugged that it is impossible to smooth them out. If this is hard to picture, know that all topological manifolds of dimension $n < 4$ can be smoothed out into smooth manifolds. The first examples of *pointy* manifolds occur in dimension 4. Historically, Kervaire's manifold was the first one discovered, which is a compact 10 dimensional topological manifold.