Point-Set Topology: Lecture 26

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1 Metrization Theorems

So far the only metrization theorem we have is Urysohn's. It makes use of normality. We know from homework that metrizable spaces are perfectly normal. In improving Urysohn's metrization theorem to one that contains necessary and sufficient conditions we need to upgrade to perfect normality. From a few weeks ago, a perfectly normal topological space is some space $(X,\,\tau)$ such that for all disjoint closed subsets $\mathcal{C},\mathcal{D}\subseteq X$ there is a continuous function $f:X\rightarrow[0,\,1]$ such that $\mathcal{C}=f^{-1}[\{\,0\,\}]$ and $\mathcal{D}=f^{-1}[\{\,1\,\}]$. Perfectly normal implies normal (we proved this) and it also implies completely normal (every subspace is also normal, we haven't proven this). The reformulation we want is in terms of G_δ sets.

Definition 1.1 (G_δ **Set**) A G_δ set in a topological space $(X,\,\tau)$ is a set $A\subseteq X$ such that there is a countable set $\mathcal{O}\subseteq\tau$ such that $A=\bigcap\mathcal{O}$. ■

Theorem 1.1. *If $(X,\,\tau)$ is a topological space, and if $\mathcal{U}\in\tau$, then $\mathcal{U}$ is a G_δ set.*

Proof. Let $\mathcal{O}=\{\,\mathcal{U}\,\}$. Then $\mathcal{O}$ is countable since it is finite, but also $\mathcal{U}=\bigcap\mathcal{O}$. So $\mathcal{U}$ is a G_δ set. □

Theorem 1.2. *If $(X,\,\tau)$ is metrizable, and if $\mathcal{C}\subseteq X$ is closed, then it is a G_δ set.*

Proof. Since $(X,\,\tau)$ is metrizable, there is a metric d that induces τ. For all $n\in\mathbb{N}$ define $\mathcal{U}_n$ via:

$$\mathcal{U}_n=\bigcup_{x\in\mathcal{C}}B_{\frac{1}{n+1}}^{(X,\,d)}(x)\tag{1}$$

Then $\mathcal{U}_n$ is open, being the union of open sets. Also $\mathcal{U}_{n+1}\subseteq\mathcal{U}_n$ and $\mathcal{C}\subseteq\mathcal{U}_n$ for all $n\in\mathbb{N}$. Suppose $x\in\bigcap_{n\in\mathbb{N}}\mathcal{U}_n$. Then for all $n\in\mathbb{N}$ there is an $a_n\in\mathcal{C}$ such that $d(x,\,a_n)<\frac{1}{n+1}$. But then $a_n\rightarrow x$. But $(X,\,\tau)$ is metrizable, and hence sequential, so since $\mathcal{C}$ is closed, if $a:\mathbb{N}\rightarrow\mathcal{C}$ is a convergent sequence with limit $x\in X$, then $x\in\mathcal{C}$. Hence $\bigcap_{n\in\mathbb{N}}\mathcal{U}_n=\mathcal{C}$ so $\mathcal{C}$ is a G_δ set. □

This is part of the idea we wish to capture. We want closed sets to be G_δ sets. Topological spaces with this property are given a name.

Definition 1.2 (G_δ **Topological Space**) A G_δ topological space is a topological space $(X,\, \tau)$ such that every closed subset $\mathcal{C} \subseteq X$ is a G_δ set. ■

Theorem 1.3. *If $(X,\, \tau_X)$ and $(Y,\, \tau_Y)$ are topological spaces, if $f : X \to Y$ is continuous, and if $A \subseteq Y$ is a G_δ set, then $f^{-1}[A]$ is a G_δ set.*

Proof. Since A is a G_δ set there is a sequence $\mathcal{V} : \mathbb{N} \to \tau_Y$ such that $A = \bigcap_{n\in\mathbb{N}} \mathcal{V}_n$. But then:

$$f^{-1}[A] = f^{-1}\Big[\bigcap_{n\in\mathbb{N}} \mathcal{V}_n\Big] = \bigcap_{n\in\mathbb{N}} f^{-1}[\mathcal{V}_n] \tag{2}$$

But since f is continuous and $\mathcal{V}_n \in \tau_Y$, we have that $f^{-1}[\mathcal{V}_n] \in \tau_X$, and hence $f^{-1}[A]$ is a G_δ set. □

Theorem 1.4. *If $(X,\, \tau)$ is a topological space, then it is perfectly normal if and only if for all closed $\mathcal{C} \subseteq X$ there is a continuous function $f : X \to [0,\, 1]$ such that $\mathcal{C} = f^{-1}[\{\,0\,\}]$.*

Proof. If $(X,\, \tau)$ is perfectly normal, let $\mathcal{D} = \emptyset$. Then $\mathcal{C}$ and $\mathcal{D}$ are disjoint closed sets so there is a continuous function $f : X \to [0,\, 1]$ such that $\mathcal{C} = f^{-1}[\{\,0\,\}]$ and $\mathcal{D} = f^{-1}[\{\,1\,\}]$. So in particular, $\mathcal{C} = f^{-1}[\{\,0\,\}]$. Now suppose $(X,\, \tau)$ is such that for all closed $\mathcal{C}$ there is a continuous function $f : X \to [0,\, 1]$ such that $\mathcal{C} = f^{-1}[\{\,0\,\}]$. Given $\mathcal{C}, \mathcal{D}$ closed and disjoint, let f be the corresponding function for $\mathcal{C}$ and g the function for $\mathcal{D}$. Define:

$$h(x) = \frac{f(x)}{f(x) + g(x)} \tag{3}$$

This is well-defined since $\mathcal{C} \cap \mathcal{D} = \emptyset$. For if $x \in \mathcal{C}$, then $f(x) = 0$ so the denominator is $g(x)$ and $g(x) > 0$ for all $x \notin \mathcal{D}$. If $x \in \mathcal{D}$ then $g(x) = 0$, so the denominator is $f(x)$ and $f(x) > 0$ for all $x \notin \mathcal{C}$. If $x \notin \mathcal{C}$ and $x \notin \mathcal{D}$ then $f(x)+g(x) > 0$. It is continuous since it is the quotient of continuous functions. Lastly, $h^{-1}[\{\,0\,\}] = \mathcal{C}$ and $h^{-1}[\{\,1\,\}] = \mathcal{D}$. For $h(x) = 0$ if and only if $f(x) = 0$ and hence $x \in \mathcal{C}$. Also $h(x) = 1$ if and only if $f(x) = f(x) + g(x)$ which is true if and only if $g(x) = 0$, meaning $x \in \mathcal{D}$. So $(X,\, \tau)$ is perfectly normal. □

Perfect normality is equivalent to normal plus G_δ. To prove this requires the topological version of one of the foundational theorems of real analysis.

Theorem 1.5. *If $(X,\, \tau)$ is a topological space, if $F : \mathbb{N} \to C(X,\, \mathbb{R})$ is a sequence of continuous functions, if $f : X \to \mathbb{R}$ is such that for all $x \in X$ it is true that $F_n(x) \to f(x)$, and if:*

$$\lim_{n\to\infty} \sup_{x\in X} |F_n(x) - f(x)| = 0 \tag{4}$$

then f is continuous.

Proof. We use the equivalent definition of continuity that for all $x \in X$ and for all $\mathcal{V} \in \tau_{\mathbb{R}}$ such that $f(x) \in \mathcal{V}$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq \mathcal{V}$. Let $x \in X$ and $\mathcal{V} \in \tau_{\mathbb{R}}$ be such that $f(x) \in \mathcal{V}$. Since $\mathcal{V}$ is open in $\mathbb{R}$ there is an $\varepsilon > 0$ such that $(f(x)-\varepsilon,\ f(x)+\varepsilon) \subseteq \mathcal{V}$. But then, since $\sup|F_n(x)-f(x)| \to 0$, there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ and for all $x \in X$ we have:

$$|F_n(x) - f(x)| < \frac{\varepsilon}{3} \tag{5}$$

But $F_N : X \to [0,\ 1]$ is continuous, so there is an open set $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $f[\mathcal{U}] \subseteq (f(x) - \varepsilon/3,\ f(x) + \varepsilon/3)$. But then, for all $x, y \in \mathcal{U}$, we have:

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - F_N(x) + F_N(x) - F_N(y) + f_N(y) - f(y)| && (6) \\ &\leq |f(x) - f_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - f(y)| && (7) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} && (8) \\ &= \varepsilon && (9) \end{aligned}$$

and hence for all $y \in \mathcal{U}$ we have $f(y) \in (f(x) - \varepsilon,\ f(x) + \varepsilon)$, so $f[\mathcal{U}] \subseteq (f(x) - \varepsilon,\ f(x) + \varepsilon)$, and hence $f[\mathcal{U}] \subseteq \mathcal{V}$. Thus, f is continuous. $\square$

Theorem 1.6. *If $(X,\ \tau)$ is a topological space, then it is perfectly normal if and only if it is normal and a G_δ space.*

Proof. Perfectly normal implies normal, we need only prove it implies G_δ as well. Let $\mathcal{C} \subseteq X$ be closed. Since $(X,\ \tau)$ is perfectly normal, there is a continuous function $f : X \to [0,\ 1]$ such that $\mathcal{C} = f^{-1}[\{\,0\,\}]$. For all $n \in \mathbb{N}$ define:

$$\mathcal{U}_n = f^{-1}\big[[0,\ \frac{1}{n+1})\big] \tag{10}$$

Since f is continuous, and since $[0,\ \frac{1}{n+1})$ is open in the subspace topology for $[0,\ 1]$, $\mathcal{U}_n$ is open. Also $\mathcal{C} \subseteq \mathcal{U}_n$ for all $n \in \mathbb{N}$. Suppose $x \in \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. Then for all $n \in \mathbb{N}$, $f(x) \in [0,\ \frac{1}{n+1})$, and hence $f(x) = 0$. But then $x \in \mathcal{C}$. Thus, $\mathcal{C} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$ so $\mathcal{C}$ is a G_δ set. That is, if $(X,\ \tau)$ is perfectly normal, then it is a normal G_δ space. In the other direction, suppose $(X,\ \tau)$ is a normal G_δ space. By a previous theorem to prove normality it suffices to show that for all closed $\mathcal{C} \subseteq X$ there is a continuous function $f : X \to [0,\ 1]$ such that $\mathcal{C} = f^{-1}[\{\,0\,\}]$. Since $(X,\ \tau)$ is a G_δ space and $\mathcal{C}$ is closed, there is a sequence $\mathcal{U} : \mathbb{N} \to \tau$ such that $\mathcal{C} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. But then for all $n \in \mathbb{N}$ we have $\mathcal{C} \subseteq \mathcal{U}_n$ and hence $\mathcal{C} \cap (X \setminus \mathcal{U}_n) = \emptyset$. But $\mathcal{U}_n$ is open, so $X \setminus \mathcal{U}_n$ is closed. By Urysohn's lemma there is a continuous function $F_n : X \to [0,\ 1]$ such that $\mathcal{C} \subseteq F_n^{-1}[\{\,0\,\}]$ and $X \setminus \mathcal{U}_n \subseteq F_n^{-1}[\{\,1\,\}]$. Define:

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{F_n(x)}{2^n} \tag{11}$$

Then since $0 \leq F_n(x) \leq 1$ for all $n \in \mathbb{N}$ and all $x \in X$, the n^{th} term of this sum is bounded by $1/2^n$, which converges to zero. By the previous theorem

f is continuous. Since $F_n(x) = 0$ for all $x \in \mathcal{C}$ and all $n \in \mathbb{N}$, we have that $f(x) = 0$ for all $x \in \mathcal{C}$. Suppose $x \notin \mathcal{C}$. Then there is a $\mathcal{U}_n$ such that $x \notin \mathcal{U}_n$, since $\mathcal{C} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$. But then $F_n(x) > 0$, and hence $f(x) > 0$. So $x \notin f^{-1}[\{0\}]$. That is, $\mathcal{C} = f^{-1}[\{0\}]$, so (X, τ) is perfectly normal. □

The key to the Nagata-Smirnov theorem is σ locally finite bases. From a few lectures ago, a σ locally finite cover of a topological space (X, τ) is a cover $\mathcal{O} \subseteq \tau$ such that there exists a sequence $\Delta : \mathbb{N} \to \mathcal{P}(\tau)$ such that for all $n \in \mathbb{N}$ it is true that $\Delta_n \subseteq \tau$ is a locally finite collection, and such that $\mathcal{O} = \bigcup_{n\in\mathbb{N}} \Delta_n$. σ locally finite basis just adds the word *basis*.

Definition 1.3 (σ **Locally Finite Basis**) A σ locally finite basis for a topological space (X, τ) is a basis $\mathcal{B}$ such that $\mathcal{B}$ is a σ locally finite open cover. ■

Theorem 1.7. *If (X, τ) is a regular topological space, if $\mathcal{B}$ is a σ locally finite bases for τ, and if $\mathcal{U} \in \tau$, then there is a sequence $\mathcal{V} : \mathbb{N} \to \tau$ such that $\mathcal{U} = \bigcup_{n\in\mathbb{N}} Cl_\tau(\mathcal{V}_n)$.*

Proof. Since $\mathcal{B}$ is a σ locally finite basis, there is a sequence $\Delta : \mathbb{N} \to \mathcal{P}(\tau)$ such that for all $n \in \mathbb{N}$ it is true that Δ_n is locally finite and such that $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \Delta_n$. For all $n \in \mathbb{N}$ define $\mathcal{A}_n$ via:

$$\mathcal{A}_n = \{\, \mathcal{W} \in \mathcal{B}_n \mid \mathrm{Cl}_\tau(\mathcal{W}) \subseteq \mathcal{U} \,\} \tag{12}$$

Since Δ_n is locally finite, so is $\mathcal{A}_n$. Define $\mathcal{V}_n$ via:

$$\mathcal{V}_n = \bigcup \mathcal{A}_n \tag{13}$$

Then $\mathcal{V}_n$ is open, being the union of open sets, and since $\mathcal{A}_n$ is locally finite we have:

$$\mathrm{Cl}_\tau(\mathcal{V}_n) = \mathrm{Cl}_\tau\Big(\bigcup \mathcal{A}_n\Big) \tag{14}$$

$$= \mathrm{Cl}_\tau\Big(\bigcup_{\mathcal{W}\in\mathcal{A}_n} \mathcal{W}\Big) \tag{15}$$

$$= \bigcup_{\mathcal{W}\in\mathcal{A}_n} \mathrm{Cl}_\tau(\mathcal{W}) \tag{16}$$

But for all $\mathcal{W} \in \mathcal{A}_n$ it is true that $\mathrm{Cl}_\tau(\mathcal{W}) \subseteq \mathcal{U}$ by definition of $\mathcal{A}_n$, and hence $\mathrm{Cl}_\tau(\mathcal{V}_n) \subseteq \mathcal{U}$. Since this is true for all $n \in \mathbb{N}$ we have:

$$\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_\tau(\mathcal{V}_n) \subseteq \mathcal{U} \tag{17}$$

But (X, τ) is regular and $\mathcal{B}$ is a basis. So given $x \in X$ there is a $\mathcal{W} \in \mathcal{B}$ such that $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{U}$. From regularity there is a $\mathcal{W}' \in \tau$ such that $x \in \mathcal{W}'$ and $\mathrm{Cl}_\tau(\mathcal{W}') \subseteq \mathcal{W}$. But again, since $\mathcal{B}$ is a basis, there is a $\mathcal{W}'' \in \mathcal{B}$ such that $x \in \mathcal{W}''$ and $\mathcal{W}'' \subseteq \mathcal{W}'$. But then $x \in \mathcal{W}''$ and $\mathrm{Cl}_\tau(\mathcal{W}'') \subseteq \mathcal{U}$. Since $\mathcal{W}'' \in \mathcal{B}$ and

$\mathcal{B} = \bigcup_{n\in\mathbb{N}} \Delta_n$ there is an $n \in \mathbb{N}$ such that $\mathcal{W}'' \in \Delta_n$. But then, by definition of $\mathcal{A}_n$, $\mathcal{W}'' \in \mathcal{A}_n$ and hence $x \in \mathcal{V}_n$. From this we obtain:

$$\mathcal{U} \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_\tau(\mathcal{V}_n) \tag{18}$$

Meaning $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_\tau(\mathcal{V}_n)$. □

Theorem 1.8. *If $(X,\,\tau)$ is a regular topological space, and if $\mathcal{B}$ is a σ locally finite basis for τ, then $(X,\,\tau)$ is a G_δ space.*

Proof. Let $\mathcal{C} \subseteq X$ be closed. Then $X \setminus \mathcal{C}$ is open. But since $(X,\,\tau)$ is regular and $\mathcal{B}$ is a σ locally finite basis, there is a sequence $\mathcal{U} : \mathbb{N} \to \tau$ such that $X \setminus \mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_\tau(\mathcal{U}_n)$. But then for all $n \in \mathbb{N}$ we have $\mathcal{C} \subseteq X \setminus \mathrm{Cl}_\tau(\mathcal{U}_n)$, and hence $\mathcal{C} \subseteq \bigcap_{n\in\mathbb{N}} \big(X \setminus \mathrm{Cl}_\tau(\mathcal{U}_n)\big)$. Let $x \in \bigcap_{n\in\mathbb{N}} \big(X \setminus \mathrm{Cl}_\tau(\mathcal{U}_n)\big)$ and suppose $x \notin \mathcal{C}$. Then $x \in X \setminus \mathcal{C}$, so there is some $n \in \mathbb{N}$ such that $x \in \mathrm{Cl}_\tau(\mathcal{U}_n)$. But then $x \notin X \setminus \mathrm{Cl}_\tau(\mathcal{U}_n)$, which is a contradiction since $x \in \bigcap_{n\in\mathbb{N}} \big(X \setminus \mathrm{Cl}_\tau(\mathcal{U}_n)\big)$. So $x \in \mathcal{C}$, and therefore $\mathcal{C}$ is the countable intersection of open sets, meaning it is a G_δ set. So all closed subsets of X are G_δ sets, meaning $(X,\,\tau)$ is a G_δ space. □

Theorem 1.9. *If $(X,\,\tau)$ is a regular topological space, and if $\mathcal{B}$ is a σ locally finite basis for τ, then $(X,\,\tau)$ is perfectly normal.*

Proof. Regularity and a σ locally finite basis implies $(X,\,\tau)$ is a G_δ space by the previous theorem. To prove the space is perfectly normal we need only prove it is normal, since $(X,\,\tau)$ is perfectly normal if and only if it is normal and a G_δ space. Let $\mathcal{C}, \mathcal{D} \subseteq X$ be disjoint closed sets. Then $X \setminus \mathcal{C}$ and $X \setminus \mathcal{D}$ are disjoint open sets. But since $(X,\,\tau)$ is regular and has a σ locally finite basis there are sequences $\mathcal{U}, \mathcal{V} : \mathbb{N} \to \tau$ such that $X \setminus \mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_\tau(\mathcal{U}_n)$ and $X \setminus \mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_\tau(\mathcal{V}_n)$. Define $\tilde{\mathcal{U}}_n$ and $\tilde{\mathcal{V}}_n$ via:

$$\tilde{\mathcal{U}}_n = \mathcal{U}_n \setminus \bigcup_{k=0}^{n} \mathrm{Cl}_\tau(\mathcal{V}_n) \tag{19}$$

$$\tilde{\mathcal{V}}_n = \mathcal{V}_n \setminus \bigcup_{k=0}^{n} \mathrm{Cl}_\tau(\mathcal{U}_n) \tag{20}$$

Each $\tilde{\mathcal{U}}_n$ and $\tilde{\mathcal{V}}_n$ is open since they are the set difference of a finite union of closed sets (which is hence closed) from an open set. Since $\mathcal{C}$ is disjoint from each $\mathrm{Cl}_\tau(\mathcal{U}_n)$ we have that $\mathcal{C} \subseteq \bigcup_{n\in\mathbb{N}} \tilde{\mathcal{U}}_n$. Similarly for $\mathcal{D}$ with $\bigcup_{n\in\mathbb{N}} \tilde{\mathcal{V}}_n$. Moreover, by the construction given, $\tilde{\mathcal{U}}_n$ and $\tilde{\mathcal{V}}_m$ are disjoint for all $m, n \in \mathbb{N}$. But then $\bigcup_{n\in\mathbb{N}} \tilde{\mathcal{U}}_n$ and $\bigcup_{n\in\mathbb{N}} \tilde{\mathcal{V}}_n$ are disjoint open sets that cover $\mathcal{D}$ and $\mathcal{C}$, respectively, so $(X,\,\tau)$ is normal. But since $(X,\,\tau)$ is also a G_δ space, we conclude that $(X,\,\tau)$ is perfectly normal. □

The Nagata-Smirnov theorem is proved in a manner similar to the Urysohn metrization theorem. For Urysohn's theorem we used the fact that $\mathbb{R}^\infty$, the countable product of $\mathbb{R}$ with itself, is metrizable metric metric d defined by:

$$d(a,\, b) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|a_n - b_n|}{1 + |a_n - b_n|} \tag{21}$$

This also gives $[0,\, 1]^{\infty}$ a metric showing the countable product of the closed unit interval is metrizable. The uncountable product of $[0,\, 1]$ is not metrizable in the product topology since it is not even first countable. We can give it a different topology, called the uniform topology, that makes it metrizable. Indeed, the topology is simply defined by a metric. Given any index set I we can topologize $[0,\, 1]^I = \prod_{\alpha \in I}[0,\, 1]$ via the metric:

$$d(\mathbf{x},\, \mathbf{y}) = \sup\Big(\{\, |x_\alpha - y_\alpha| \mid \alpha \in I \,\}\Big) \tag{22}$$

This topology does not need to be the product topology nor the box topology. The usefulness comes from the fact that it gives us a metric on arbitrarily large products. We prove the Nagata-Smirnov theorem by showing any metrizable space can be embedded as a subspace of $[0,\, 1]^I$ for some index set I equipped with the uniform topology.

Theorem 1.10 (**The Nagata-Smirnov Metrization Theorem**). *If $(X,\, \tau)$ is regular, Hausdorff, and has a σ locally finite basis $\mathcal{B}$, then it is metrizable.*

Proof. Since $(X,\, \tau)$ is regular and has a σ locally finite basis, it is perfectly normal. But for all $\mathcal{U} \in \mathcal{B}$, since $\mathcal{B}$ is a basis, $\mathcal{U}$ is open, so $X \setminus \mathcal{U}$ is closed. But since $(X,\, \tau)$ is perfectly normal there is a continuous function $\tilde{f}_{\mathcal{U}} : X \to [0,\, 1]$ such that $X \setminus \mathcal{U} = f^{-1}[\{\, 0 \,\}]$. For each $n \in \mathbb{N}$ define $f_{n,\mathcal{U}} : X \to [0,\, \frac{1}{n+1}]$ via:

$$f_{n,\mathcal{U}}(x) = \frac{1}{n+1} \tilde{f}_{\mathcal{U}}(x) \tag{23}$$

Then $f_{n,\mathcal{U}}^{-1}[\{\, 0 \,\}] = X \setminus \mathcal{U}$ as well since this is just a scaling of $\tilde{f}_{\mathcal{U}}$ by a non-zero constant. Define I via:

$$I = \{\, f_{n,\mathcal{U}} \mid \mathcal{U} \in \mathcal{B},\, n \in \mathbb{N} \,\} \tag{24}$$

I can be indexed by $\mathbb{N} \times \mathcal{B}$ since $f_{n,\mathcal{U}}$ corresponds to the ordered pair $(n,\, \tilde{f}_{\mathcal{U}})$. We define $F : X \to [0,\, 1]^I$ via:

$$F(x) = \alpha \text{ where } \alpha_{(n,\mathcal{U})} = f_{n,\mathcal{U}}(x) \tag{25}$$

We need to show F is an embedding. That is, it is a homeomorphism onto its image. First, F is injective. If $x, y \in X$, $x \neq y$, then since $(X,\, \tau)$ is Hausdorff, there are open sets $\mathcal{V}_x, \mathcal{V}_y \in \tau$ such that $x \in \mathcal{V}_x$, $y \in \mathcal{V}_y$, and $\mathcal{V}_x \cap \mathcal{V}_y = \emptyset$. But $\mathcal{B}$ is a basis so there is $\mathcal{U}_x, \mathcal{U}_y \in \mathcal{B}$ such that $x \in \mathcal{U}_x$, $\mathcal{U}_x \subseteq \mathcal{V}_x$, and $y \in \mathcal{U}_y$, $\mathcal{U}_y \subseteq \mathcal{V}_y$. But then $f_{0,\mathcal{U}_x}(x) \neq 0$ and $f_{0,\mathcal{U}_x}(y) = 0$, so the $(0,\, \mathcal{U})$ component of $F(x)$ and $F(y)$ differ, hence $F(x) \neq F(y)$. So F is injective. The function F is an embedding with respect to the product topology, and the uniform topology is finer, meaning $F : X \to F[X]$ is an open mapping. All that's left to prove is that F is continuous. But $[0,\, 1]^I$ is a metrizable space when given the uniform topology, so we just need to show for all $x \in X$ and for all $\varepsilon > 0$ there is an open set $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and for all $y \in \mathcal{U}$ we have $d(F(x),\, F(y)) < \varepsilon$. Since $\mathcal{B}$ is σ locally finite there are countably many sets Δ_n, each of which is locally finite, such that $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \Delta_n$. Given $x \in X$ there is then an open set $\mathcal{U}_n$ such that $x \in \mathcal{U}_n$ and only finitely many elements of Δ_n intersect $\mathcal{U}_n$. But each $f_{n,\,\mathcal{V}}$ is continuous, so of the finitely many functions $f_{n,\,\mathcal{V}}$ with $\mathcal{V} \in \Delta_n$ where $\mathcal{U}_n \cap \mathcal{V} \neq \emptyset$ we can find an open set $\mathcal{W}_n$ such that each function varies by at most $\varepsilon/2$. Let $N \in \mathbb{N}$ be such that $N > 2/\varepsilon$. Let $\mathcal{U}$ be defined by:

$$\mathcal{U} = \bigcap_{k=0}^{N} \mathcal{W}_k \tag{26}$$

If $n \in \mathbb{N}$ and $n \leq N$, then by how $\mathcal{U}$ is defined we have, for all $\mathcal{V} \in \mathcal{B}$ and all $y \in \mathcal{U}$, the following:

$$|f_{n,\,\mathcal{V}}(x) - f_{n,\,\mathcal{V}}(y)| < \frac{\varepsilon}{2} \tag{27}$$

If $n > N$ then, since $f_{n,\,\mathcal{V}}$ has co-domain $[0,\, \frac{1}{n+1}]$, we have:

$$|f_{n,\,\mathcal{V}}(x) - f_{n,\,\mathcal{V}}(y)| \leq \frac{1}{n+1} + \frac{1}{n+1} = \frac{2}{n+1} < \varepsilon \tag{28}$$

But then, for all $y \in \mathcal{U}$, we have:

$$d\big(F(x),\, F(y)\big) = \sup\Big(\big\{\, |f_{n,\,\mathcal{V}}(x) - f_{n,\,\mathcal{V}}(y)| \;\big|\; (n,\, \mathcal{V}) \in I \,\big\}\Big) < \varepsilon \tag{29}$$

and hence for all $y \in \mathcal{U}$ we have $d(F(x),\, F(y)) < \varepsilon$, so F is continuous. So $(X,\, \tau)$ is homeomorphic to a subspace of $[0,\, 1]^I$ with the uniform topology, which is metrizable, and hence $(X,\, \tau)$ is metrizable. □

The converse of this theorem is true as well. The next metrization theorem is Smirnov's. It uses paracompactness in place of σ locally finite bases, and also the property of *local* metrizability.

Definition 1.4 (**Locally Metrizable Topological Space**) A locally metrizable topological space is a topological space $(X,\, \tau)$ such that for all $x \in X$ there is a $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$ and $(\mathcal{U},\, \tau_{\mathcal{U}})$ is metrizable, where $\tau_{\mathcal{U}}$ is the subspace topology. ■

Theorem 1.11 (**The Smirnov Metrization Theorem**). *If $(X,\, \tau)$ is paracompact, locally metrizable, and Hausdorff, then it is metrizable.*

The converse of this is true as well. Metrizable implies Hausdorff and locally metrizable. The only hard part is that metrizable implies paracompact, and this is Stone's theorem.