Point-Set Topology: Lecture 27

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1 Tychonoff's Theorem

We conclude our study of compactness and paracompactness with Tychonoff's theorem which states that the product of any collection $(X_\alpha,\,\tau_\alpha)$, $\alpha \in I$, of compact spaces, equipped with the product topology, is still compact. Note this is not true if the product is given the box topology. The definition of the product topology is crucial for this theorem. In part because the product topology has as a subbasis the collection of all sets $\prod_{\alpha\in I}\mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = X_\alpha$ for all but at most one $\alpha \in I$.

The proof requires, in some form, the axiom of choice. Indeed, Tychonoff's theorem is equivalent to the axiom of choice, given the other axioms of set theory. The easiest presentation of the theorem is a quick corollary of Alexander's subbasis theorem, which uses Zorn's lemma in it's proof. Zorn's lemma, which is also equivalent to the axiom of choice, says the following:

Theorem 1.1 (**Zorn's Lemma**). *If $(X,\,\leq)$ is a partially ordered set, meaning $\leq$ is transitive, anti-symmetric, and symmetric, such that for every chain $A \subseteq X$, which is a subset such that $(A,\,\leq_A)$ is totally ordered, is bounded, then there is a maximal element $a \in X$ which is an element such that for all $b \in X$, $a \leq b$ implies $b = a$.*

With this, one can prove Alexander's theorem.

Theorem 1.2 (**Alexander's Subbasis Theorem**). *If $(X,\,\tau)$ is a topological space, and if $\mathcal{B} \subseteq \tau$ is a subbasis of τ that covers X such that for all open covers $\mathcal{O} \subseteq \mathcal{B}$ there is a finite subcover $\Delta \subseteq \mathcal{O}$, then $(X,\,\tau)$ is compact.*

Note the formulation of this theorem is not quite the definition of compactness. Compactness says you need to look at every open cover and find a finite subcover. The theorem says it is sufficient to look at open covers that consist only of elements from the subbasis $\mathcal{B}$.

Theorem 1.3 (**Tychonoff's Theorem**). *If I is a set such that for all $\alpha \in I$ the ordered pair $(X_\alpha,\,\tau_\alpha)$ is a compact topological space, then $(\prod_{\alpha\in I}X_\alpha,\,\tau_{\prod})$ is compact where $\tau_{\prod}$ is the product topology.*