Point-Set Topology: Lecture 28

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1 Topological Manifolds

Topological manifolds are spaces that *look* like $\mathbb{R}^n$, and that are topologically *nice*. They are one of the primary motivators for general topology and have widespread applications in physics, computer graphics, and other branches of mathematics.

Definition 1.1 (**Locally Euclidean Topological Space**) A locally Euclidean topological space is a topological space $(X,\,\tau)$ such that for all $x\in X$ there is an open set $\mathcal{U}\in\tau$ such that $x\in\mathcal{U}$, an $n\in\mathbb{N}$, and a continuous injective open mapping $\varphi:\mathcal{U}\rightarrow\mathbb{R}^n$. ■

Since $\varphi:\mathcal{U}\rightarrow\mathbb{R}^n$ is an injective function, it is bijective onto it's image. Hence $\varphi:\mathcal{U}\rightarrow\varphi[\mathcal{U}]$ is a continuous bijective open mapping, which is therefore a homeomorphism. Another way of stating the definition of locally Euclidean spaces is that every point has an open set about it that is homeomorphic to an open subset of $\mathbb{R}^n$ for some $n\in\mathbb{N}$.

Example 1.1 Euclidean space $\mathbb{R}^n$ with the standard Euclidean topology $\tau_{\mathbb{R}^n}$ is locally Euclidean. Given a point $\mathbf{x}\in\mathbb{R}^n$ choose $\mathcal{U}=\mathbb{R}^n$ and $\varphi=\mathrm{id}_{\mathbb{R}^n}$. That is, the open set about $\mathbf{x}$ is all of Euclidean space, and the function φ is the identity. The identity is always a homeomorphism, which is therefore a continuous injective open mapping. This shows us that Euclidean space is indeed locally Euclidean. ■

Example 1.2 If $\mathcal{V}\subseteq\mathbb{R}^n$ is an open subset, then $(\mathcal{V},\,\tau_{\mathcal{V}})$ is locally Euclidean where $\tau_{\mathcal{V}}$ is the subspace topology inherited from the standard Euclidean topology $\tau_{\mathbb{R}^n}$. Given a point $\mathbf{x}\in\mathcal{V}$, choose $\mathcal{U}=\mathcal{V}$ and $\varphi=\iota_{\mathcal{V}}$, the inclusion mapping. Since $\mathcal{V}$ is open, $\iota_{\mathcal{V}}$ is an open mapping, and inclusion mappings are always injective and continuous. Thus any open subspace of $\mathbb{R}^n$ is locally Euclidean. ■

Example 1.3 The bug-eyed line, which is the quotient of $\mathbb{R}\times\mathbb{Z}_2$ under the identification $(x,\,0)R(x,\,1)$ for all $x\neq 0$, is locally Euclidean. For points away from the double-origin the bug-eyed line looks, locally, like the real line. The only cause for concern is the two origins. Label $0'=[(0,\,0)]$ and $0''=[(0,\,1)]$. The set $(-1,\,1)\times\{\,0\,\}$ is open in the product space $\mathbb{R}\times\mathbb{Z}_2$ since $\mathbb{Z}_2$ is discrete and $(-1,\,1)$ is open in $\mathbb{R}$. The set $\big((-1,\,0)\cup(0,\,1)\big)\times\{\,1\,\}$ is also open for the same reason. Let $\mathcal{U}$ be the union of these two open sets, which is therefore open. Note that this set is saturated with respect to the quotient map. That is, $\mathcal{U} = q^{-1}\big[q[\mathcal{U}]\big]$. But the quotient map takes saturated open sets to saturated open sets, so $q[\mathcal{U}]$ is open in the quotient topology. This open set is homeomorphic to $(-1,\,1)$ and contains $0'$. We can do a similar argument for $0''$ showing that the bug-eyed line is locally Euclidean. Note, however, that it is not Hausdorff. ■

Example 1.4 The long-line is locally Euclidean. Given any point in the long-line it *locally* looks like $(-1,\,1)$, the open unit interval in $\mathbb{R}$. This space is very *large*, it is not second-countable. Note then that is would be impossible to embed the long-line into Euclidean space, no matter how large the dimension. ■

Topological manifolds should be nice enough that it is possible to embed them into $\mathbb{R}^n$. This is not a requirement, however, but the previous two examples show us that *locally Euclidean* by itself can still yield bizarre examples. This motivates the following definition.

Definition 1.2 (**Topological Manifold**) A topological manifold is a topological space $(X,\,\tau)$ that is locally Euclidean, second-countable, and Hausdorff. ■

Example 1.5 $\mathbb{R}^n$ is a topological manifold, with its standard topology. It is locally Euclidean, as seen previously, and it is also second-countable (being the product of a second-countable space) and Hausdorff (since it is metrizable). ■

Example 1.6 If $\mathcal{V} \subseteq \mathbb{R}^n$ is an open subset, then $(\mathcal{V},\,\tau_{\mathcal{V}})$ is a topological manifold. We've already shown that such a space is locally Euclidean, but it is also Hausdorff and second-countable since these properties are inherited by subspaces. ■

Example 1.7 As far as set theory is concerned, a function $f : A \to B$ from a set A to a set B is a subset of $A \times B$ satisfying certain properties. We can use this to define locally Euclidean topological spaces by looking at continuous functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ for some $m, n \in \mathbb{N}$. Given $f : \mathbb{R}^m \to \mathbb{R}^n$, continuous, $f \subseteq \mathbb{R}^m \times \mathbb{R}^n$ can be given the subspace topology. This makes it a closed subset since f is continuous. It is also a locally Euclidean subspace. For given $\big(\mathbf{x},\, f(\mathbf{x})\big) \in f$, let $\mathcal{U} = f$ and define $F : f \to \mathbb{R}^m$ via:

$$F\big((\mathbf{x},\, f(\mathbf{x})\big) = \mathbf{x} \tag{1}$$

This is just the projection of the elements of $f \subseteq \mathbb{R}^m \times \mathbb{R}^n$ onto $\mathbb{R}^m$. Projections are continuous. Let's show F is injective and an open mapping. It is injective since given:

$$\big(\mathbf{x}_0,\, f(\mathbf{x}_0)\big) \neq \big(\mathbf{x}_1,\, f(\mathbf{x}_1)\big) \tag{2}$$

we must have $\mathbf{x}_0 \neq \mathbf{x}_1$ (since if $\mathbf{x}_0 = \mathbf{x}_1$, then $f(\mathbf{x}_0) = f(\mathbf{x}_1)$ by definition of a function). So then:

$$F\big((\mathbf{x}_0,\, f(\mathbf{x}_0)\big) \neq F\big((\mathbf{x}_1,\, f(\mathbf{x}_1)\big) \tag{3}$$

meaning F is injective. There is a continuous inverse $F^{-1} : \mathbb{R}^m \to f$ given by:

$$F^{-1}(\mathbf{x}) = \big(\mathbf{x},\, f(\mathbf{x})\big) \tag{4}$$

Since f is continuous, F^{-1} is continuous since both components are continuous. So F is an open mapping and f is a locally Euclidean subspace of $\mathbb{R}^m \times \mathbb{R}^n$. Since subspaces of $\mathbb{R}^{m+n}$ are also Hausdorff and second-countable, this shows us that the *graph* of a continuous function $f : \mathbb{R}^m \to \mathbb{R}^n$ is a topological manifold. ■

Example 1.8 $\mathbb{S}^1$ with the subspace topology from $\mathbb{R}^2$ is locally Euclidean. We'll show this in two ways. First, via orthographic projection. We split the circle into four parts:

$$\mathcal{U}_{\text{North}} = \{\,(x,\,y) \in \mathbb{S}^1 \mid y > 0\,\} \tag{5}$$
$$\mathcal{U}_{\text{South}} = \{\,(x,\,y) \in \mathbb{S}^1 \mid y < 0\,\} \tag{6}$$
$$\mathcal{U}_{\text{East}} = \{\,(x,\,y) \in \mathbb{S}^1 \mid x > 0\,\} \tag{7}$$
$$\mathcal{U}_{\text{West}} = \{\,(x,\,y) \in \mathbb{S}^1 \mid x < 0\,\} \tag{8}$$

See Fig. 1. Then we define four functions:

$$\varphi_{\text{North}} : \mathcal{U}_{\text{North}} \to \mathbb{R} \qquad \varphi_{\text{North}}\big((x,\,y)\big) = x \tag{9}$$
$$\varphi_{\text{South}} : \mathcal{U}_{\text{South}} \to \mathbb{R} \qquad \varphi_{\text{South}}\big((x,\,y)\big) = x \tag{10}$$
$$\varphi_{\text{East}} : \mathcal{U}_{\text{East}} \to \mathbb{R} \qquad \varphi_{\text{East}}\big((x,\,y)\big) = y \tag{11}$$
$$\varphi_{\text{West}} : \mathcal{U}_{\text{West}} \to \mathbb{R} \qquad \varphi_{\text{West}}\big((x,\,y)\big) = y \tag{12}$$

Since these are projection mappings, they are continuous. From how the four open sets are defined, each is also injective. To show it is an open mapping we just need to find a continuous inverse with respect to the image of these sets. Note that for all four functions the range is $(-1,\,1)$. We have the following inverse functions:

$$\varphi_{\text{North}}^{-1}(x) = \big(x,\,\sqrt{1-x^2}\big) \tag{13}$$
$$\varphi_{\text{South}}^{-1}(x) = \big(x,\,-\sqrt{1-x^2}\big) \tag{14}$$
$$\varphi_{\text{East}}^{-1}(y) = \big(\sqrt{1-y^2},\,y\big) \tag{15}$$
$$\varphi_{\text{West}}^{-1}(y) = \big(-\sqrt{1-y^2},\,y\big) \tag{16}$$

each of which is continuous since the square root function is continuous. The four sets also cover $\mathbb{S}^1$, showing that $\mathbb{S}^1$ is locally Euclidean. Since $\mathbb{R}^2$ is second-countable and locally Euclidean, $\mathbb{S}^1$ is as well. Hence the circle is a topological manifold. ■

This shows we can cover $\mathbb{S}^1$ using four sets each of which is homeomorphic to an open subset of $\mathbb{R}$. We can do better, only two sets are needed. Place an observer at the north pole $N = (0,\,1)$. Given any other point $(x,\,y)$ the line from the observer to the point is not parallel to the x axis, meaning eventually it must intersect it. Let's solve for when. The line segment $\alpha(t) = (1-t)N + t(x,\,y)$ starts at the north pole at time $t = 0$ and ends at the point $(x,\,y)$ on the circle at time $t = 1$. The line intersects the x axis when the y component is zero.

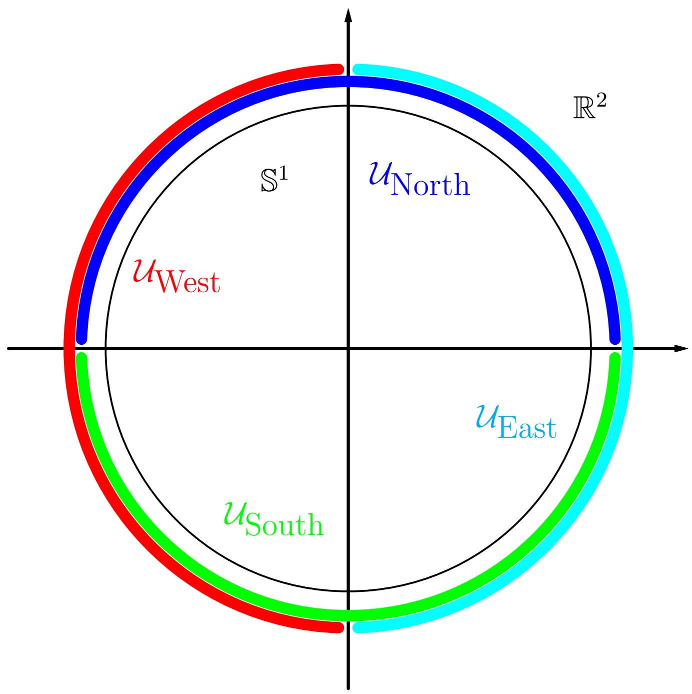

Figure 1: Cover of $\mathbb{S}^1$ with Locally Euclidean Sets

Thus we wish to solve $1 - t + ty = 0$ for t. We get:

$$t_0 = \frac{1}{1-y} \tag{17}$$

The x coordinate at time $t = t_0$ is then:

$$\varphi_N\big((x,\, y)\big) = \frac{x}{1-y} \tag{18}$$

This is *stereographic projection* about the north pole. It is continuous since it is a rational function. It is also bijective with a continuous inverse. Given $X \in \mathbb{R}$ we can solve for the value $(x,\, y) \in \mathbb{S}^1$ that gets mapped to X by reversing the previous process. The line $\beta(t) = (1-t)N + t(X,\, 0)$ starts at the north pole and ends at $(X,\, 0)$. We wish to solve for the time t when $||\beta(t)||_2 = 1$ which corresponds to the moment the line intersects the circle. We have:

$$\begin{aligned}
||\beta(t)||_2 &= ||(1-t)N + t(X,\, 0)||_2 && (19)\\
&= ||(1-t)(0,\, 1) + t(X,\, 0)||_2 && (20)\\
&= ||(tX,\, 1-t)||_2 && (21)\\
&= \sqrt{(tX)^2 + (1-t)^2} && (22)
\end{aligned}$$

Solving for $||\beta(t)||_2 = 1$ is equivalent to solving $||\beta(t)||_2^2 = 1$ so we need to consider the expression $(tX)^2 + (1-t)^2$. We get:

$$\begin{aligned}
1 &= (tX)^2 + (1-t)^2 && (23)\\
&= t^2X^2 + 1 - 2t + t^2 && (24)\\
&= t^2(1 + X^2) - 2t + 1 && (25)
\end{aligned}$$

meaning we want to solve for $t^2(1+X^2) - 2t = 0$. The solution $t = 0$ corresponds to the North pole, which is not the one we want. Dividing through by t we get:

$$t_1 = \frac{2}{1+X^2} \tag{26}$$

The point $(x,\, y)$ corresponds to $\beta(t_1)$ and is given by:

$$\varphi_N^{-1}(X) = \Big(\frac{2X}{1+X^2},\, \frac{-1+X^2}{1+X^2}\Big) \tag{27}$$

This function is continuous since it is a rational function in each component. Because of this $\varphi_N : \mathbb{S}^1 \setminus \{\, (0,\, 1)\,\} \rightarrow \mathbb{R}$ is a homeomorphism. Doing a similar projection about the south pole shows that $\mathbb{S}^1$ can be covered by two open sets, $\mathbb{S}^1 \setminus \{\, (0,\, 1)\,\}$ and $\mathbb{S}^1 \setminus \{\, (0,\, -1)\,\}$, each of which is homeomorphic to $\mathbb{R}$.

It is impossible to do this with one set. This is because $\mathbb{S}^1$ is not homeomorphic to an open subset of $\mathbb{R}$ since $\mathbb{S}^1$ is compact and the only open subset of $\mathbb{R}$ that is compact is the empty set, but $\mathbb{S}^1$ is non-empty. So two is the best we can do.

Example 1.9 The sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is also locally Euclidean for all $n \in \mathbb{N}$. Define $\mathcal{U}_k^{\pm} \subseteq \mathbb{S}^n$ via:

$$\mathcal{U}_k^+ = \{\, \mathbf{x} \in \mathbb{S}^n \mid \mathbf{x}_k > 0 \,\} \tag{28}$$

$$\mathcal{U}_k^- = \{\, \mathbf{x} \in \mathbb{S}^n \mid \mathbf{x}_k < 0 \,\} \tag{29}$$

These $2n+2$ open sets cover $\mathbb{S}^n$ and each is homeomorphic to an open subset of $\mathbb{R}^n$. Define $\varphi_k^{\pm} : \mathcal{U}_k^{\pm} \to B_1^{\mathbb{R}^n}(\mathbf{0})$ via:

$$\varphi_k^{\pm}(\mathbf{x}) = (\mathbf{x}_0,\, \dots,\, \mathbf{x}_{k-1},\, \mathbf{x}_{k+1},\, \mathbf{x}_n) \tag{30}$$

That is, projecting down that k^{th} axis. This is continuous with a continuous inverse ${\varphi_k^{\pm}}^{-1} : B_1^{\mathbb{R}^n}(\mathbf{0}) \to \mathcal{U}_k^{\pm}$ given by:

$${\varphi_k^{\pm}}^{-1}(\mathbf{x}) = (\mathbf{x}_0,\, \dots,\, \mathbf{x}_{k-1},\, \pm\sqrt{1 - ||\mathbf{x}||_2^2},\, \mathbf{x}_k,\, \dots,\, \mathbf{x}_{n-1}) \tag{31}$$

This is also continuous, so $\mathbb{S}^n$ is locally Euclidean. For reasons similar to the circle, the higher dimensional spheres are also topological manifolds. ■

These mappings are called *orthographic projections*. They are formed by placing an observer at *infinity* and projecting what they see down to the plane. This is shown in Fig. 2

Definition 1.3 (**Topological Chart**) A topological chart of dimension n in a topological space $(X,\, \tau)$ about a point $x \in X$ is an ordered pair $(\mathcal{U},\, \varphi)$ such that $\mathcal{U} \in \tau$, $x \in \mathcal{U}$, and $\varphi : \mathcal{U} \to \mathbb{R}^n$ is an injective continuous open mapping. ■

Locally Euclidean could equivalently be described as a topological space $(X,\, \tau)$ such that for all $x \in X$ there is a chart $(\mathcal{U},\, \varphi)$ such that $x \in \mathcal{U}$. A collection of charts that covers a space is called an *atlas*. See Fig. 3.

Definition 1.4 (**Topological Atlas**) A topological atlas for a topological space $(X,\, \tau)$ is a set $\mathcal{A}$ of topological charts in $(X,\, \tau)$ such that for all $x \in X$ there is a $(\mathcal{U},\, \varphi) \in \mathcal{A}$ such that $x \in \mathcal{U}$. ■

That is, an atlas is a collection of charts whose domains cover the space. Think of an actual atlas used for navigating. The pages consist of various locations on the globe, but only provides local information. To get information that is more global requires piecing some of the charts of the atlas together. A locally Euclidean space is a topological space $(X,\, \tau)$ such that there exists an atlas $\mathcal{A}$ for it. We've shown that $\mathbb{S}^n$ can be covered by $2n+2$ charts using orthographic projection. We can do better using stereographic projection the same way we did for $\mathbb{S}^1$. This is shown for $\mathbb{S}^2$ in Fig. 4.

There are two other types of projections that are useful for geometric reasons in covering $\mathbb{S}^n$. These are the near-sided and far-sided projections. Near-sided projection is shown in Fig. 5. The idea is to take an observer and place them somewhere on the z axis above the sphere. The portion of the sphere that

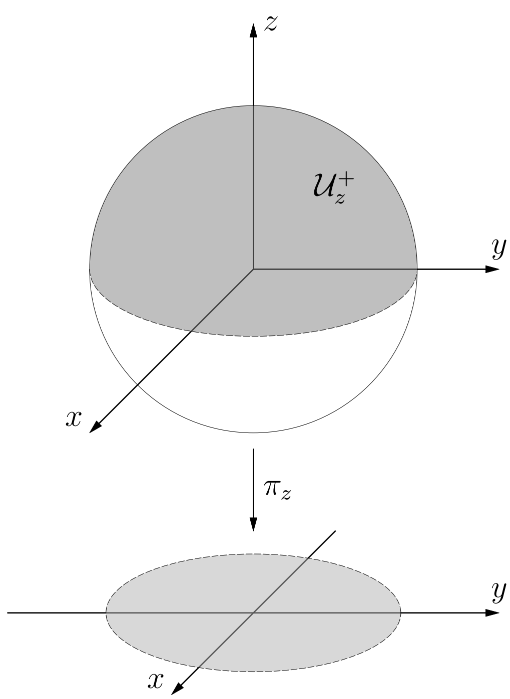

Figure 2: Orthographic Projection of the Sphere

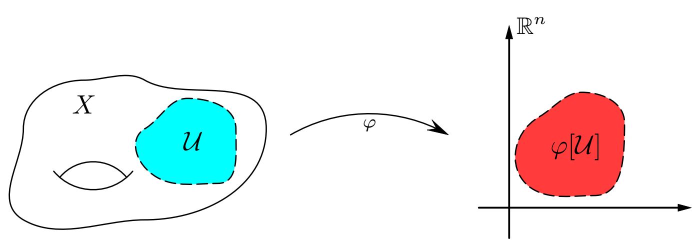

Figure 3: A Chart in a Manifold

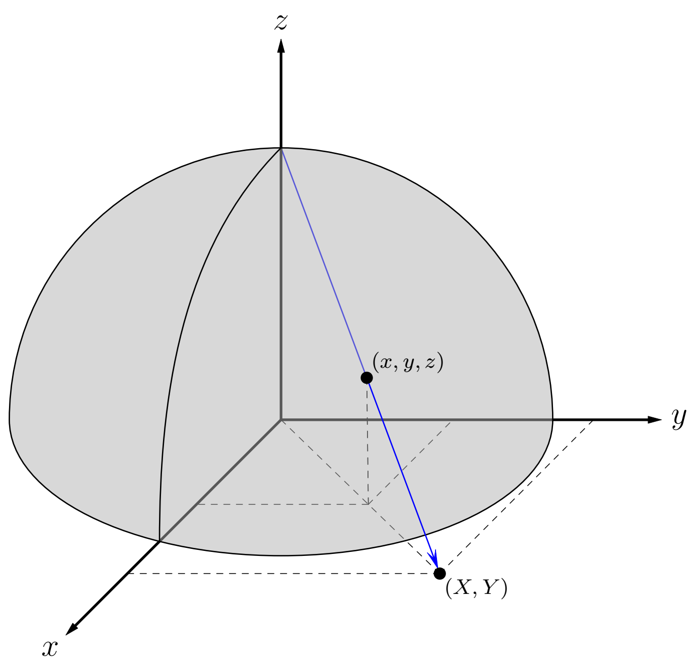

Figure 4: Stereographic Projection for the Sphere

is visible is then projected down to the xy plane. Far-sided projection is the opposite. You place the observer at the same spot but remove everything that can be seen. The result is a hollow semi-sphere. You then unwrap this on to the plane to get the projection. This is shown in Fig. 6. Stereographic projection is then just far-sided projection at the north pole, and orthographic projection is near-sided projection at infinity.

Example 1.10 (**Real Projective Space**) Let $X = \mathbb{R}^{n+1} \setminus \{ \mathbf{0} \}$. Define the equivalence relation R on X via $\mathbf{x}R\mathbf{y}$ if and only if $\mathbf{y} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R} \setminus \{ 0 \}$. $\mathbb{RP}^n$ is the set X/R and the topology $\tau_{\mathbb{RP}^n}$ is the quotient topology induced by R. As a set this is the set of all lines in $\mathbb{R}^{n+1}$ that pass through the origin. That is, a point $[\mathbf{x}] \in \mathbb{RP}^n$ is the entire line through the origin that passes through the point $\mathbf{x}$. Let's start with $\mathbb{RP}^1$. Any line can be described by an angle $0 \leq \theta < \pi$. If you vary the line you are on slightly, you are just varying this angle. Hopefully it becomes intuitive that $\mathbb{RP}^1$ is in fact a one dimensional locally Euclidean space (it may not be intuitive as to why it is Hausdorff or second countable, but we'll get there). A similar thinking applies to $\mathbb{RP}^n$. Let's be precise. Let $\mathcal{U}_k \subseteq X$ be defined by:

$$\mathcal{U}_k = \{ \mathbf{x} \in \mathbb{R}^{n+1} \setminus \{ \mathbf{0} \} \mid \mathbf{x}_k \neq 0 \} \tag{32}$$

This is the complement of the k^{th} axis, which is open since the k^{th} axis is closed. It is also saturated with respect to the canonical quotient map $q : X \to \mathbb{RP}^n$

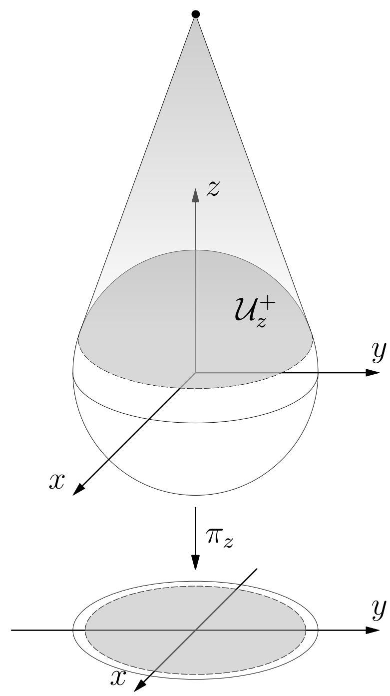

Figure 5: Near-Sided Projection of the Sphere

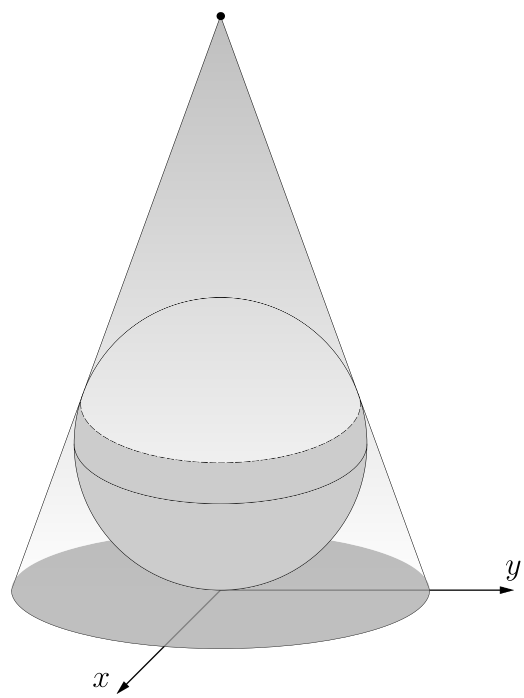

Figure 6: Far-Sided Projection of the Sphere

defined by $q(\mathbf{x}) = [\mathbf{x}]$. That is, $q^{-1}\big[q[\mathcal{U}_k]\big] = \mathcal{U}_k$. It is always the case that $\mathcal{U}_k \subseteq q^{-1}\big[q[\mathcal{U}_k]\big]$, let's show this reverses for our particular set $\mathcal{U}_k$. Let $\mathbf{x} \in q^{-1}\big[q[\mathcal{U}_k]\big]$. Then $[\mathbf{x}] \in q\big[\mathcal{U}_k\big]$ so there is some $\mathbf{y} \in \mathcal{U}_k$ such that $[\mathbf{x}] = [\mathbf{y}]$. But then $\mathbf{y}_k \neq 0$ and $\mathbf{x} = \lambda\mathbf{y}$ for some $\lambda \in \mathbb{R} \setminus \{\,0\,\}$. But then $\mathbf{x}_k \neq 0$, and hence $\mathbf{x} \in \mathcal{U}_k$. So $\mathcal{U}_k$ is saturated. But since q is a quotient map, if $\mathcal{U}_k$ is open and saturated, the set $\tilde{\mathcal{U}}_k = q[\mathcal{U}_k]$ is open. Define $\varphi_k : \tilde{\mathcal{U}}_k \to \mathbb{R}^n$ via:

$$\varphi_k\big([\mathbf{x}]\big) = \Big(\frac{\mathbf{x}_0}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_{k-1}}{\mathbf{x}_k},\,\frac{\mathbf{x}_{k+1}}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_n}{\mathbf{x}_k}\Big) \tag{33}$$

We have to prove this is well-defined in two regards. First, there is no division by zero since $\mathbf{x} \in \mathcal{U}_k$ implies $\mathbf{x}_k \neq 0$. Second, this is well defined as a function. By that I mean if $[\mathbf{x}] = [\mathbf{y}]$, then there is some $\lambda \in \mathbb{R} \setminus \{\,0\,\}$ such that $\mathbf{y} = \lambda\mathbf{x}$. But then:

$$\varphi_k\big([\mathbf{y}]\big) = \Big(\frac{\mathbf{y}_0}{\mathbf{y}_k},\,\dots,\,\frac{\mathbf{y}_{k-1}}{\mathbf{y}_k},\,\frac{\mathbf{y}_{k+1}}{\mathbf{y}_k},\,\dots,\,\frac{\mathbf{y}_n}{\mathbf{y}_k}\Big) \tag{34}$$

$$= \Big(\frac{\lambda\mathbf{x}_0}{\lambda\mathbf{x}_k},\,\dots,\,\frac{\lambda\mathbf{x}_{k-1}}{\lambda\mathbf{x}_k},\,\frac{\lambda\mathbf{x}_{k+1}}{\lambda\mathbf{x}_k},\,\dots,\,\frac{\lambda\mathbf{x}_n}{\lambda\mathbf{x}_k}\Big) \tag{35}$$

$$= \Big(\frac{\mathbf{x}_0}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_{k-1}}{\mathbf{x}_k},\,\frac{\mathbf{x}_{k+1}}{\mathbf{x}_k},\,\dots,\,\frac{\mathbf{x}_n}{\mathbf{x}_k}\Big) \tag{36}$$

$$= \varphi_k\big([\mathbf{x}]\big) \tag{37}$$

So it is well-defined. It is also continuous. This is one of the characteristics of the quotient map. Given a topological space $(Y,\,\tau_Y)$ and a function $f : X/R \to Y$, f is continuous if and only if $f \circ q : X \to Y$ is continuous where $q : X \to X/R$ is the canonical quotient map. The composition $\varphi_k \circ q$ is a rational function, which is continuous, so φ_k is continuous. The inverse function is given by:

$$\varphi_k^{-1}(\mathbf{x}) = \big[(\mathbf{x}_0,\,\dots,\,\mathbf{x}_{k-1},\,1,\,\mathbf{x}_k,\,\dots,\,\mathbf{x}_{n-1})\big] \tag{38}$$

which is continuous since the function $f : \mathbb{R}^n \to \mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\}$ defined by:

$$f(\mathbf{x}) = (\mathbf{x}_0,\,\dots,\,\mathbf{x}_{k-1},\,1,\,\mathbf{x}_k,\,\dots,\,\mathbf{x}_{n-1}) \tag{39}$$

is continuous, so φ_k^{-1} is the composition of continuous functions. Since the sets $\mathcal{U}_k$ cover $\mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\}$, the sets $\tilde{\mathcal{U}}_k$ also cover $\mathbb{RP}^n$. Because of this $\mathbb{RP}^n$ is locally Euclidean. It is also second countable since it can be covered with finitely many open sets each of which is homeomorphic to an open subset of $\mathbb{R}^n$, which is hence second countable. Since $\mathbb{RP}^n$ is the finite union of second countable open subspaces, it is second countable itself. It is also Hausdorff. Given $[\mathbf{x}] \neq [\mathbf{y}]$ we have that $\mathbf{y}$ is not of the form $\lambda\mathbf{x}$ for any real number, meaning $\mathbf{x}$ and $\mathbf{y}$ lie on different lines through the origin. Let θ be defined by:

$$\theta = \frac{1}{4}\arccos\Big(\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||_2\,||\mathbf{y}||_2}\Big) \tag{40}$$

θ is one-fourth the angle made between the lines through the origin spanned by $\mathbf{x}$ and $\mathbf{y}$. Let $\mathcal{U}$ and $\mathcal{V}$ be defined by:

$$\mathcal{U} = \big\{\,\mathbf{z} \in \mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\} \mid \measuredangle(\mathbf{x},\,\mathbf{z}) < \theta\,\big\} \tag{41}$$

$$\mathcal{V} = \big\{\,\mathbf{z} \in \mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\} \mid \measuredangle(\mathbf{y},\,\mathbf{z}) < \theta\,\big\} \tag{42}$$

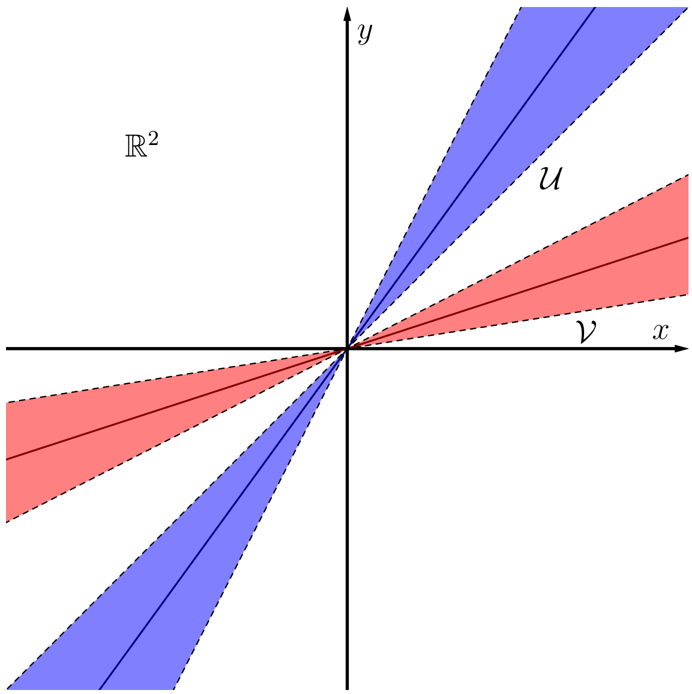

Figure 7: $\mathbb{RP}^n$ is Hausdorff

Where $\measuredangle(\mathbf{p}, \mathbf{q})$ is the angle between the non-zero vectors $\mathbf{p}$ and $\mathbf{q}$. These sets are open cones in $\mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\}$ (Fig. 7) which are also saturated with respect to q, and by the choice of θ they are disjoint. But then $\tilde{\mathcal{U}} = q[\mathcal{U}]$ and $\tilde{\mathcal{V}} = q[\mathcal{V}]$ are disjoint open subsets of $\mathbb{RP}^n$ such that $[\mathbf{x}] \in \tilde{\mathcal{U}}$ and $[\mathbf{y}] \in \tilde{\mathcal{V}}$. Hence $\mathbb{RP}^n$ is Hausdorff. The real projective space is therefore a topological manifold. ■

The elements of $\mathbb{RP}^n$ are equivalence classes of $\mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\}$. A *point* in $\mathbb{RP}^n$ is a *line* in $\mathbb{R}^{n+1}$ through the origin. It is not immediately clear that $\mathbb{RP}^n$ can be embedded as a subspace of $\mathbb{R}^N$ for some $N \in \mathbb{N}$. It indeed can, in fact $\mathbb{RP}^n$ can be embedded into $\mathbb{R}^{2n}$ for all $n > 0$, but this is by no means obvious. The case $n = 1$ is slightly obvious if you really think about what $\mathbb{RP}^1$ is (it's just a circle $\mathbb{S}^1$). The case $\mathbb{RP}^2$ is less obvious ($\mathbb{RP}^2$ is **not** a sphere). We can not embed the real projective plane into $\mathbb{R}^3$, unlike the sphere. If we try we'll end up with a surface that must intersect itself. This is shown in Fig. 8. This representation is known as the *cross cap*. We can do better than this. The cross cap has a crease in it, and this can be removed. David Hilbert, one of the pioneering mathematicians of the early 20th century, thought it impossible to draw the real projective plane in $\mathbb{R}^3$ in such a way that it has no crease. He asked his student Werney Boy to try and prove this. Instead Boy discovered a method of drawing the real projective plane in $\mathbb{R}^3$ that has no crease (it is still self intersecting). This is called the *Boy surface*. It is shown in Fig. 9. Bryant and

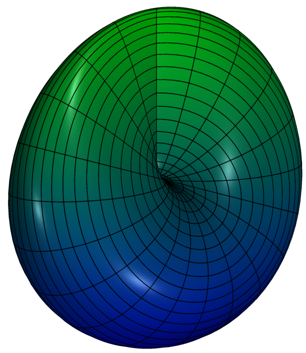

Figure 8: The Real Projective Plane

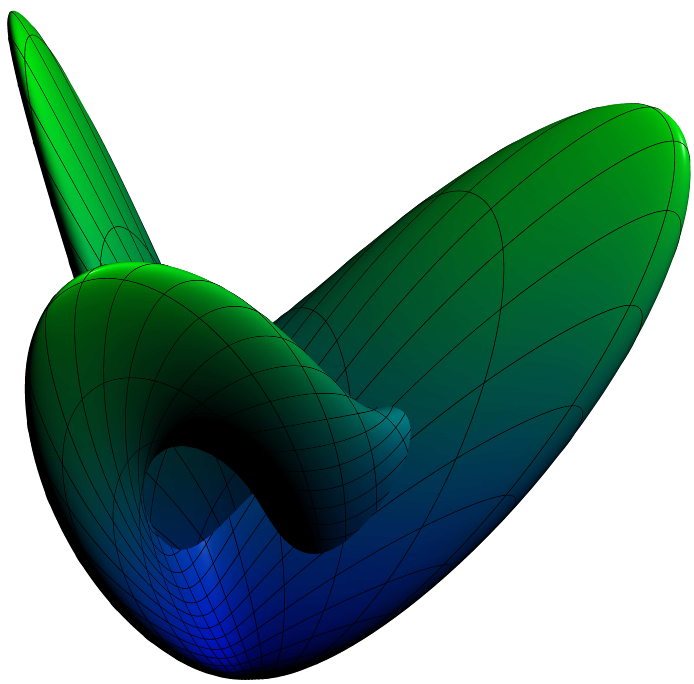

Figure 9: The Boy Surface

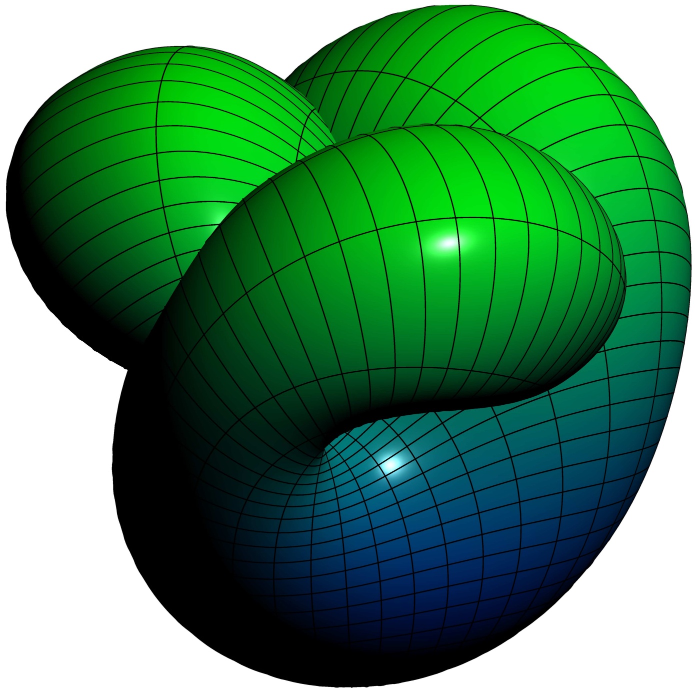

Figure 10: The Bryant-Kusner Parameterization of the Boy Surface

Kusner discovered a way to do this using somewhat simpler functions involving complex variables. The Bryant-Kusner parameteriation is shown in Fig. 10.

2 Smooth Manifolds

General topological spaces do not have a notion of derivative. We can speak of continuity, but differentiation in $\mathbb{R}^n$ requires a function to *locally* be approximated by a *tangent-hyperplane*. That is, the tangent line for functions $f:\mathbb{R}\rightarrow\mathbb{R}$, and tangent plane for functions $F:\mathbb{R}^2\rightarrow\mathbb{R}$. *Topological vector spaces* (vector spaces with a topology that makes scalar multiplication and vector addition continuous) have enough structure, but this is perhaps too prohibitive. Topological manifolds also have enough structure that one can ask questions about smoothness. Let (X,τ) be a topological manifold, and let $x\in X$. Given two charts $(\mathcal{U},\varphi)$ and $(\mathcal{V},\psi)$ that contain x, the function $\psi\circ\varphi^{-1}$ is a continuous function from $\varphi[\mathcal{U}\cap\mathcal{V}]$ to $\psi[\mathcal{U}\cap\mathcal{V}]$, both of which are open subsets of $\mathbb{R}^n$. That is, if we label $\mathcal{E}=\varphi[\mathcal{U}\cap\mathcal{V}]$ and $\mathcal{W}=\psi[\mathcal{U}\cap\mathcal{V}]$, then $\psi\circ\varphi^{-1}:\mathcal{E}\rightarrow\mathcal{W}$ is a continuous function from an open subset of Euclidean space to another open subset of Euclidean space. It is then perfectly valid to ask if this function has partial derivatives, or continuous partial derivatives, or if *all partial derivatives of all orders* exist. That is, if the function is *smooth.*

This motivates the following.

Definition 2.1 (**Smoothly Compatible Charts**) Smoothly compatible charts in a topological space (X, τ) are charts $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ in (X, τ) such that either $\mathcal{U} \cap \mathcal{V} = \emptyset$, or the function $\psi \circ \varphi^{-1} : \varphi[\mathcal{U} \cap \mathcal{V}] \to \psi[\mathcal{U} \cap \mathcal{V}]$ is smooth (as a function from on open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$). ■

Definition 2.2 (**Smooth Atlas**) A smooth atlas on a topological space (X, τ) is an atlas $\mathcal{A}$ on (X, τ) such that for all $(\mathcal{U}, \varphi), (\mathcal{V}, \psi) \in \mathcal{A}$ it is true that $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ are *smoothly compatible*. ■

Recall our analogy with an actual atlas. A smooth atlas says that as you transition from one chart to another, this is done *smoothly*. That is, suppose you have two maps containing sections of Europe. The first map has Paris, the second map has Berlin, and they overlap somewhere in between. As you travel from Paris to Berlin you start with the first map, since it contains Paris. Eventually you'll reach the edge of the first map and need to start using the second. When you do this, when you *transition* between maps, it would be nice if the second map looked roughly the same as the first map in this overlapping region. That is, it would be nice if you could *smoothly transition* between maps. This is precisely what the smooth compatibility condition does. A smooth atlas allows you to navigate anywhere in your space using smooth transitions.

Definition 2.3 (**Smooth Manifold**) A smooth manifold is an ordered triple $(X, \tau, \mathcal{A})$ such that (X, τ) is a topological manifold, and $\mathcal{A}$ is a smooth atlas for (X, τ). ■

The examples of topological manifolds that we've so far discussed are all smooth manifolds, when the appropriate atlas is chosen. This then inspires the following question.

$$\text{Are all topological manifolds also smooth manifolds?} \tag{43}$$

Let's provide a definition.

Definition 2.4 (**Smoothable Manifold**) A smoothable manifold is a topological manifold (X, τ) such that there exists a smooth atlas $\mathcal{A}$ for (X, τ). ■

Kervaire showed in 1960 that there are 10-dimensional topological manifolds that are not smoothable. That is, there is no smooth atlas for the space. Freedman found a compact 4-dimensional example in 1982. Kuiper's example, found in 1967, gives an 8-dimensional example that has a rather explicit formula. Note that the sphere $\mathbb{S}^2$ can be described by $x^2 + y^2 = 1$. Under certain conditions, functions $f : X \to \mathbb{R}^n$ can yield topological manifolds by considering $f^{-1}[\{\,\mathbf{c}\,\}]$ for some constant $\mathbf{c} \in \mathbb{R}^n$. For example $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$. The circle is the pre-image $f^{-1}[\{\,1\,\}]$. In a similar manner, consider the function $f : \mathbb{C}^5 \to \mathbb{C}$ defined by:

$$\begin{aligned} &f(z_1, z_2, z_3, z_4, z_5) = \\ &\quad z_1^5(1 + z_1) + z_2^3(1 + e z_2^3) + z_3^2(1 + e^2 z_3^4) + z_4^2(1 + e^3 z_4^4) + z_5^2(1 + e^4 z_5^4) \end{aligned} \tag{44}$$

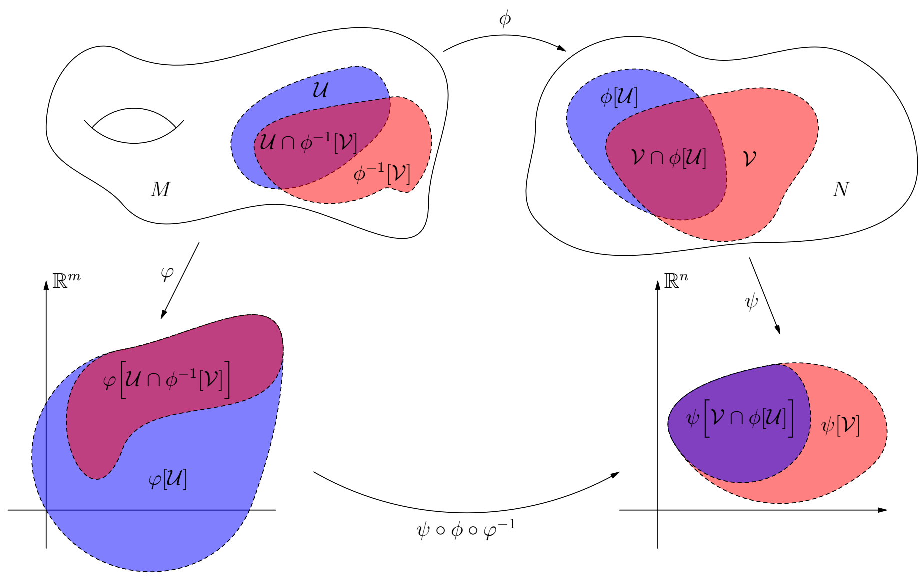

Figure 11: Smooth Function Between Manifolds

Where e is the standard Euler constant. Note that $\mathbb{C}^5$ can be identified with $\mathbb{R}^{10}$ and $\mathbb{C}$ can be seen as $\mathbb{R}^2$. The pre-image of $\{ 0 \}$ is an 8-dimensional manifold that cannot be smoothed.

In dimensions 1, 2, and 3, all topological manifolds are smoothable. The cube may not be smooth itself, but we can smooth it out into a sphere. Similarly, a triangle can be smoothed into a circle. The fact that some topological manifolds cannot be smooth becomes very hard to imagine. This difficulty is amplified by the fact that the first examples of such a phenomenon occur with four dimensional spaces.

3 Smooth Functions and Tangent Spaces

Given two smooth manifolds $(X, \tau_X, \mathcal{A}_X)$ and $(Y, \tau_Y, \mathcal{A}_Y)$, it is possible to speak of *smooth* functions between them. The key here is to locally translate the problem back to $\mathbb{R}^n$ and ask about smoothness there. We use the transition maps to do this. Note that if $\phi : X \to Y$ is a function, if $x \in X$, and if $(\mathcal{U}, \varphi) \in \mathcal{A}_X$ and $(\mathcal{V}, \psi) \in \mathcal{A}_Y$ are charts such that $x \in \mathcal{U}$ and $f(x) \in \mathcal{V}$, then the function $\psi \circ \phi \circ \varphi^{-1}$ is a function from a subset of $\mathbb{R}^m$ to a subset of $\mathbb{R}^n$. If ϕ is *continuous*, then this composition is a function between an open subset of $\mathbb{R}^m$ and a subset of $\mathbb{R}^n$ and we can ask questions about smoothness. The visual for this scheme is provided in Fig. 11. This motivates the following definition.

Definition 3.1 (**Smooth Functions Between Smooth Manifolds**) A smooth function from a smooth manifold $(X, \tau_X, \mathcal{A}_X)$ to a smooth manifold $(Y, \tau_Y, \mathcal{A}_Y)$ is a continuous function $\phi : X \to Y$ such that for all $x \in X$ there are charts $(\mathcal{U}, \varphi) \in \mathcal{A}_X$ and $(\mathcal{V}, \psi) \in \mathcal{A}_Y$ such that $x \in \mathcal{U}$, $f(x) \in \mathcal{V}$, and the composition $\psi \circ \phi \circ \varphi^{-1}$ is a smooth function from an open subset of $\mathbb{R}^m$ to a subset of $\mathbb{R}^n$. ■

Note the definition does not require $m = n$. The two smooth manifolds can have different dimensions. You could, for example, smoothly embed the circle $\mathbb{S}^1$ into the sphere $\mathbb{S}^2$ by placing it along the equator.

Apart from allowing us to perform calculus with manifolds, smooth functions also allow us to define the notion of *tangent space*. For surfaces in $\mathbb{R}^3$ we define the tangent space to a point as a plane that contains point and lies *tangential* to the surface. That is, it is the best linear approximation to the surface near the point. This definition requires an embedding of the surface into $\mathbb{R}^3$, whereas the general smooth manifold is an abstract topological space and has no embedding into $\mathbb{R}^N$ associated with it. It is indeed possible to embed a smooth manifold of dimension n into some Euclidean space $\mathbb{R}^N$ ($N = 2n + 1$ does the trick by Whitney's theorem), but this takes some work. Given an n dimensional smooth manifold $(X, \tau, \mathcal{A})$ it would be nice if we can define *tangent spaces* without extra assumptions about embeddings. We do this using *derivations*.

Let $C^\infty(X, \mathbb{R})$ denote the set of all *smooth* functions $f : X \to \mathbb{R}$ where $\mathbb{R}$ carries the standard Euclidean smooth structure. A derivation at a point $x \in X$ is a function $D : C^\infty(X, \mathbb{R}) \to \mathbb{R}$ such that:

$$D(af + bg) = aD(f) + bD(g) \qquad \text{(Linearity)}$$
$$D(fg) = f(x)D(g) + D(f)g(x) \qquad \text{(Liebniz Rule)}$$

This second condition is the product rule from calculus. The tangent space at x is defined as follows.

Definition 3.2 (**Tangent Space in a Smooth Manifold**) The tangent space to a point $x \in X$ in a smooth manifold $(X, \tau_X, \mathcal{A}_X)$ is the set T_xX defined by:

$$T_xX = \{\, D : C^\infty(X, \mathbb{R}) \to \mathbb{R} \mid D \text{ is a derivation at } x \,\} \tag{45}$$

That is, the set of all linear and Liebnizean functions at x. ■

The tangent space at x has the structure of a vector space since derivations can be added. Since we wish to think of the elements of T_xX as *tangent vectors* starting at the point x, we usually denote them by $v \in T_xX$ or $w \in T_xX$. We can add them by using the following rule. Given a function $f \in C^\infty(X, \mathbb{R})$, we define $(v + w)(f)$ via:

$$(v + w)(f) = v(f) + w(f) \tag{46}$$

This addition on the right hand side makes sense since $v(f)$ and $w(f)$ are real numbers. Scalar multiplication is also defined:

$$(a \cdot v)(f) = a \cdot v(f) \tag{47}$$

where $a \in \mathbb{R}$ is any scalar. Being a vector space, T_xX has a dimension. It is quite fortunate (and perhaps the reason why we have defined T_xX this way) that the dimension is precisely the dimension of the manifold. This can be proved by finding an explicit basis. Pick a chart $(\mathcal{U},\,\varphi) \in \mathcal{A}$ that contains x and define the *differentiation operators* $\partial\varphi_k : C^\infty(X,\,\mathbb{R}) \to \mathbb{R}$ via:

$$\partial\varphi_k(f) = \frac{\partial}{\partial x_k}\Big|_{\mathbf{x}=\varphi(x)}\big(f \circ \varphi^{-1}\big) \tag{48}$$

Since $\varphi : \mathcal{U} \to \mathbb{R}^n$ and $f : X \to \mathbb{R}$ are smooth, we have that $f \circ \varphi^{-1}$ is a smooth function from an open subset of Euclidean space to the real numbers. Taking partial derivatives is thus well-defined. *Any* tangent vector (i.e., any derivation) $v \in T_xX$ can be written as a linear combination of these n partial derivatives. We have thus attached to every point $x \in X$ a real-vector space T_xX, mimicing the notion of *tangent space* for surfaces in $\mathbb{R}^3$. We've also accomplished this without embedding the manifold into Euclidean space.

4 Riemannian Manifolds

Topological manifolds belong to point-set (or *general*) topology. Smooth manifolds initiate the study of *differential topology.* Geometry starts when we can measure things like lengths and angles, volumes and areas, and so-on. Smooth manifolds have no means of making such measurements. To do so we need an *angle measuring device.* In $\mathbb{R}^n$ this is given by the Euclidean dot product. We define:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=0}^{n-1} x_k y_k \tag{49}$$

From this we may define lengths.

$$||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \tag{50}$$

and from this we obtain angles.

$$\angle(\mathbf{x},\,\mathbf{y}) = \arccos\Big(\frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}\Big) \tag{51}$$

Geometry begins with the ability to define a *dot-product* on the tangent vectors of a smooth manifold. This is given via a *Riemannian metric.* The rest of these notes attempt to define this notion.

A *vector field* V on a smooth manifold $(X,\,\tau,\,\mathcal{A})$ is an assignment of a tangent vector $v \in T_xX$ to every $x \in X$. That is, at every point in the space we choose an *arrow* that starts at that point. If $v \in T_xX$ is the tangent vector assigned to $x \in X$ we denote this by $v = V_x$.

Given such a vector field, if $f \in C^\infty(X, \mathbb{R})$ is a smooth function, we obtain another function $Vf : X \to \mathbb{R}$ via:

$$(Vf)(x) = V_x(f) \tag{52}$$

Remember that V_x is a tangent vector at x, meaning it is a derivation $v : C^\infty(X, \mathbb{R}) \to \mathbb{R}$. That is, it takes in smooth functions and returns a real number. Because of this the above equation is well-defined. The vector field is said to be *smooth* if Vf is a smooth function for all $f \in C^\infty(X, \mathbb{R})$.

A *Riemannian metric* is an assigment to every point $x \in X$ a function $g_x : T_xX \times T_xX \to \mathbb{R}$ that mimics the definition of inner products. That is:

$$\begin{aligned} g_x(a_0v_0 + a_1v_1,\, w) &= a_0g_x(v_0,\, w) + a_1g_x(v_1,\, w) && \text{(Linearity)}\\ g_x(v,\, w) &= g_x(w,\, v) && \text{(Symmetry)}\\ g_x(v,\, v) &\geq 0 && \text{(Positivity)}\\ g_x(v,\, v) &= 0 \Leftrightarrow v = 0 && \text{(Definiteness)} \end{aligned}$$

Moreover, this assignment is done *smoothly*. That is, for every pair of smooth vector fields V, W on X, the function $g(V,\, W) : X \to \mathbb{R}$ defined by:

$$\big(g(V,\, W)\big)(x) = g_x(V_x,\, W_x) \tag{53}$$

is smooth. A *Riemannian manifold* is a smooth manifold together with a Riemannian metric.

While not every topological manifold is *smoothable*, it is true that every smooth manifold can be made into a Riemannian manifold. This is one of the more important applications of partitions of unity and paracompactness. Every smooth manifold is paracompact, and hence every open cover has a subordinate partition of unity. Moreover, for smooth manifolds the partition of unity can be chosen to consist solely of *smooth* functions. This new fact can be used to build a Riemannian metric on any smooth manifold.