Point-Set Topology: Lecture 30

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1 Topological Groups

Now we introduce some topology into our algebra.

Definition 1.1 (**Topological Group**) A topological group is an ordered triple $(X,\,\tau,\,*)$ where $(X,\,\tau)$ is a topological space and $(X,\,*)$ is a group such that the functions $m : X \times X \to X$ and $\eta : X \to X$ defined by:

$$m(x,\,y) = x * y \tag{1}$$

$$\eta(x) = x^{-1} \tag{2}$$

are continuous (here $X \times X$ is given the product topology). That is, the group operations are continuous functions. ■

Example 1.1 The real line with addition is a topological group. The addition of real numbers is indeed a continuous operation, and the inverse operation is negation: $x \mapsto -x$. ■

Example 1.2 More generally, $\mathbb{R}^n$ as a vector space with vector addition becomes a topological group when endowed with the standard Euclidean topology. ■

Example 1.3 The circle $\mathbb{S}^1$ with the subspace topology and the *rotation operation* is a topological group. That is, Given points $e^{i\theta},\,e^{i\phi} \in \mathbb{S}^1$, we define $e^{i\theta} * e^{i\phi} = e^{i(\theta+\phi)}$. This operation is continuous with respect to the subspace topology the circle inherits from $\mathbb{R}^2$, giving us a topological group. ■

Example 1.4 The integers $\mathbb{Z}$ with the subspace topology from $\mathbb{R}$ and addition form a topological group. Note that the subspace topology on $\mathbb{Z}$ is also the discrete topology. ■

Example 1.5 More generally, if $(X,\,*)$ is any group, then $(X,\,\mathcal{P}(X),\,*)$ is a topological group. The product topology on $X \times X$ is also the discrete topology, and hence *any* function $f : X \times X \to X$ is continuous. Similarly, any function $g : X \to X$ is continuous. So in particular the multiplication and inversion operations are continuous and $(X,\,\mathcal{P}(X),\,*)$ is a topological group. ■

Example 1.6 Going the other way, if $(X, *)$ is any group, and if $\tau = \{\emptyset, X\}$ is the indiscrete topology, then $(X, \tau, *)$ is a topological group. Since τ is the indiscrete topology, any function into X is continuous, so in particular multiplication and inversion are continuous. ■

Example 1.7 If $X = \mathbb{R}$ and $\tau = \{\emptyset, \mathbb{R}\}$ is the indiscrete topology, then by the previous example $(\mathbb{R}, \tau, +)$ is a topological group. Note that it is a non-Hausdorff topological group. Topological groups need not satisfy any of the separation properties. ■

Theorem 1.1. *If (X, τ) is a topological space, and if $(X, *)$ is a group, then $(X, \tau, *)$ is a topological group if and only if the function $f : X \times X \to X$ defined by $f(x, y) = x * y^{-1}$ is continuous.*

Proof. The function $f(x, y) = x*y^{-1}$ can be seen as a combination of multiplication and inversion. If $(X, \tau, *)$ is a topological group, then this function is continuous. In the other direction, if this function is continuous, then setting $x = e$, the identity, we have $f(e, y) = e * y^{-1} = y^{-1}$, and this is a continuous function of y, meaning inversion is continuous. But then $x * y = x * (y^{-1})^{-1} = f(x, y^{-1})$, which is the composition of continuous functions, so multiplication is continuous. Hence $(X, \tau, *)$ is a topological group. □

Theorem 1.2. *If $(X, \tau, *)$ is a topological group, and if $a \in X$, and if $L_a : X \to X$ is left-translation of X by a, then L_a is a homeomorphism.*

Proof. We have already proved that left-translation in a group is bijective. Let us show that it is continuous. But $L_a(x) = a*x$ is the restriction of $m : X \times X \to X$, defined by $m(x, y) = x * y$, to the subset $\{a\} \times X$. But the restriction of a continuous function to subspace is continuous, and hence $L_a : X \to X$ is continuous. The inverse function is given by $L_a^{-1} = L_{a^{-1}}$ since:

$$
\begin{aligned}
(L_a \circ L_{a^{-1}})(x) &= L_a\big(L_{a^{-1}}(x)\big) && \text{(Definition of Composition)}\\
&= L_a(a^{-1} * x) && \text{(Definition of } L_{a^{-1}}\text{)}\\
&= a * (a^{-1} * x) && \text{(Definition of } L_a\text{)}\\
&= (a * a^{-1}) * x && \text{(Associativity)}\\
&= e * x && \text{(Inverse)}\\
&= x && \text{(Identity)}
\end{aligned}
$$

And hence $L_a \circ L_{a^{-1}}$ is the identity function. Similarly, $L_{a^{-1}} \circ L_a$ is the identity. So the inverse of left-translation is another left-translation, which is continuous. Hence L_a is a homeomorphism. □

Two immediate results are often of equal use.

Theorem 1.3. *If $(X, \tau, *)$ is a topological group, if $a \in X$, and if $L_a : X \to X$ is left-translation by a, then L_a is an open map.*

Proof. Left-translation is a homeomorphism, so it is also an open map. □

Theorem 1.4. *If $(X, \tau, *)$ is a topological group, if $a \in X$, and if $L_a : X \to X$ is left-translation by a, then L_a is an closed map.*

Proof. Left-translation is a homeomorphism, so it is also an closed map. □

Theorem 1.5. *If $(X, \tau, *)$ is a topological group, if $a \in X$, and if $R_a : X \to X$ denotes right-translation by a, then R_a is a homeomorphism. In particular it is both an open map and a closed map.*

Proof. The proof is similar to that for left-translation. □

Conjugation is also a homeomorphism, but it is also a group isomorphism. Functions that are both continuous and homomorphisms are one of the main objects of study in topological groups.

Definition 1.2 (**Topological Group Homomorphism**) A topological group homomorphism from a topological group $(X, \tau_X, *_X)$ to a topological group $(Y, \tau_Y, *_Y)$ is a function $\varphi : X \to Y$ such that φ is continuous with respect to the topologies and also a group homomorphism with respect to the group operations. ■

For groups, a bijective homomorphism automatically yields a group isomorphism since the inverse function will be a group homomorphism. Continuity lacks such niceties. We must be careful in our defining of topological group isomorphisms.

Definition 1.3 (**Topological Group Isomorphism**) A topological group isomorphism from a topological group $(X, \tau_X, *_X)$ to another topological group $(Y, \tau_Y, *_Y)$ is a function $\varphi : X \to Y$ such that φ is a homeomorphism with respect to the topologies and also a group isomorphism with respect to the group operations. ■

Note we did **not** just define this as a *continuous group isomorphism* since we would like the inverse function to be continuous as well. Hence we defined this as a *homeomorphic group isomorphism.*

Theorem 1.6. *If $(X, \tau, *)$ is a topological group and if $g \in X$, then conj_g is a topological group isomorphism, where $\mathrm{conj}_g : X \to X$ is the conjugation function.*

Proof. We have already proven that conjugation is a group isomorphism. It is also a homeomorphism since:

$$\begin{aligned}
\mathrm{conj}_g(a) &= g * a * g^{-1} && \text{(Definition of } \mathrm{conj}_g\text{)}\\
&= g * (a * g^{-1}) && \text{(Associativity)}\\
&= L_g(a * g^{-1}) && \text{(Definition of } L_g\text{)}\\
&= L_g\big(R_{g^{-1}}(a)\big) && \text{(Definition of } R_{g^{-1}}\text{)}\\
&= (L_g \circ R_{g^{-1}})(a) && \text{(Definition of Composition)}
\end{aligned}$$

And hence conjugation is the composition of a left-translation and a right-translation, which is the composition of homeomorphisms, which is therefore a homeomorphism. □

Left-translations, right-translations, and conjugations allow us to show that topological groups have a lot of nice properties. For one thing, topological groups are *homogeneous.*

Definition 1.4 (**Homogeneous Topological Space**) A homogeneous topological space is a topological space $(X,\,\tau)$ such that for all $x, y \in X$ there is a homeomorphism $f : X \to X$ such that $f(x) = y$. ■

This means that every point in the space looks like every other point. Euclidean spaces are examples of homogeneous topological spaces, as are all connected topological manifolds. Topological groups are also homogeneous.

Theorem 1.7. *If $(X,\,\tau,\,*)$ is a topological group, then $(X,\,\tau)$ is a homogeneous topological space.*

Proof. For let $a, b \in X$ and define $f : X \to X$ by $f(x) = b * x * a^{-1}$. Then f is a homeomorphism, being the composition of left and right-translations. But also:

$$f(a) = b * a * a^{-1} = b * e = b \tag{3}$$

And hence $(X,\,\tau)$ is a homogeneous topological space. □

Homogeneity can be used to prove a lot of nice properties. For one, if there are nice *local* topological properties around the origin, then these properties might become *global* by using translations. Let's motivate this by example.

Definition 1.5 (**Kolmogorov Topological Space**) A Kolmogorov topological space is a topological space $(X,\,\tau)$ such that for all $x, y \in X$ there is an open set $\mathcal{U} \in \tau$ such that either $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, or $x \notin \mathcal{U}$ and $y \in \mathcal{U}$. ■

This is the weakest of the separation properties, weaker than the Hausdorff and Frechét properties. A Kolmogorov space need not be Hausdorff or Frechét.

Example 1.8 Define τ on $\mathbb{N}$ to be:

$$\tau = \{\, \mathbb{Z}_n \subseteq \mathbb{Z} \mid n \in \mathbb{N} \,\} \cup \{\, \mathbb{N} \,\} \tag{4}$$

This is a topology since the sets are nested, so the union and intersection properties are satisfied. It is Kolmogorov. Given $n, m \in \mathbb{N}$, $n \neq m$, choose $\mathcal{U} = \mathbb{Z}_{k+1}$ where $k = \min(m,\,n)$. Then $\mathcal{U}$ is open can contains one one of m and n, but not both. The space is not Frechét since no point can be separated from 0. ■

What's remarkable is that Kolmorogov topological groups are Frechét, Hausdorff, regular, and completely regular. We'll prove the first two of the assertions. First we'll need a little lemma.

Theorem 1.8. *If $(G,\,*)$ is a group, if $A \subseteq G$, if $x, y \in G$, if $x \in A$ and $y \notin A$, and if $B = L_x\big[R_y[A^{-1}]\big]$, where:*

$$A^{-1} = \{\, a^{-1} \in G \mid a \in A \,\} \tag{5}$$

then $x \notin B$ and $y \in B$.

Proof. First, $y \in B$ since $x \in A$, and hence $x^{-1} \in A^{-1}$, so $L_x\big(R_y(x^{-1})\big) \in B$. But:

$$L_x\big(R_y(x^{-1})\big) = x * x^{-1} * y = e * y = y \tag{6}$$

and therefore $y \in B$. Second, $x \notin B$. For if $x \in B$, then $x = x * a^{-1} * y$ for some $a \in A$. But then by the cancellation law, $a^{-1} * y = e$. But inverses are unique, meaning $y = a$. But then $y \in A$, which is a contradiction. So $x \notin B$ and $y \in B$, completing the proof. □

Theorem 1.9. *If $(X,\,\tau,\,*)$ is a topological group, and if $\eta : X \to X$ is defined by $\eta(x) = x^{-1}$, then η is a homeomorphism.*

Proof. η is continuous since $(X,\,\tau,\,*)$ is a topological group. It is also bijective. It is injective since $x^{-1} = y^{-1}$ implies $x = y$ since inverses are unique. It is surjective since $(X,\,*)$ is a group, and hence every element has an inverse. Lastly, the inverse is continuous since the inverse of η is η. That is:

$$\eta^2(x) = (\eta \circ \eta)(x) \tag{7}$$
$$= \eta\big(\eta(x)\big) \tag{8}$$
$$= \eta(x^{-1}) \tag{9}$$
$$= (x^{-1})^{-1} \tag{10}$$
$$= x \tag{11}$$

And hence $\eta = \eta^{-1}$. But then η^{-1} is continuous, so η is a homeomorphism. □

Theorem 1.10. *If $(X,\,\tau,\,*)$ is a topological group, and if $(X,\,\tau)$ is a Kolmogorov topological space, then it is a Frechét topological space.*

Proof. For let $x, y \in X$. Since $(X,\,\tau)$ is a Kolmogorov space, there is a $\mathcal{U} \in \tau$ such that either $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, or $x \notin \mathcal{U}$ and $y \in \mathcal{U}$. Suppose the former (the proof is symmetric). Let $\mathcal{V} = (L_x \circ R_y)[\mathcal{U}^{-1}]$ where:

$$\mathcal{U}^{-1} = \{\, a^{-1} \in X \mid a \in \mathcal{U} \,\} \tag{12}$$

Then $\mathcal{U}^{-1}$ is the image of the inversion function $\eta : X \to X$ defined by $\eta(x) = x^{-1}$. Since η is a homeomorphism, it is an open mapping. Since $\mathcal{U}$ is open, $\mathcal{U}^{-1}$ is also open. Hence $\mathcal{V}$ is also open since left-translations and right-translations are open mappings as well. By a previous theorem, since $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, we have $x \notin \mathcal{V}$ and $y \in \mathcal{V}$. Hence $(X,\,\tau)$ is a Frechét space. □

The Hausdorff property will be implied as well.

Theorem 1.11. *If $(X,\,\tau)$ is a topological space, then it is Hausdorff if and only if the set:*

$$\Delta = \{\, (x,\,x) \in X \times X \mid x \in X \,\} \tag{13}$$

is closed with respect to the product topology.

Proof. First, suppose Δ is closed with respect to the product topology. Let's show that (X, τ) is Hausdorff. Let $x, y \in X$ with $x \neq y$. Since $x \neq y$ we have that $(x, y) \notin \Delta$, meaning $(x, y) \in X \times X \setminus \Delta$. But since $\Delta \subseteq X \times X$ is closed, $X \times X \setminus \Delta$ is open. Then from the definition of the product topology, there must be open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $\mathcal{U} \times \mathcal{V} \subseteq X \times X \setminus \Delta$ and $(x, y) \in \mathcal{U} \times \mathcal{V}$. But also $\mathcal{U} \cap \mathcal{V} = \emptyset$. For if $z \in \mathcal{U} \cap \mathcal{V}$, then $(z, z) \in X \times X \setminus \Delta$, which is a contradiction since $(z, z) \in \Delta$. Hence (X, τ) is Hausdorff. In the other direction, suppose (X, τ) is a Hausdorff space and let's show that Δ is closed. To do this we prove that $X \times X \setminus \Delta$ is open. Let $(x, y) \in X \times X \setminus \Delta$. Then $x \neq y$. But (X, τ) is Hausdorff so there are open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$. But then $\mathcal{U} \times \mathcal{V}$ is an open subset of $X \times X$ that contains (x, y), and since the sets are disjoint the product is contained entirely inside of $X \times X \setminus \Delta$. Hence, Δ is closed. $\square$

Theorem 1.12. *If $(X, \tau, *)$ is a topological group, and if (X, τ) is a Kolmogorov space, then it is Hausdorff.*

Proof. We have proven that Kolmogorov topological groups are Frechét, and hence $\{ e \}$ is closed, where e is the unique identity element of $(X, *)$. But since $(X, \tau, *)$ is a topological group, the function $f : X \times X \to X$ defined by $f(x, y) = x * y^{-1}$ is continuous. But the set:

$$\Delta = \{ (x, x) \in X \times X \mid x \in X \} \tag{14}$$

is the pre-image of the set $\{ e \}$ by f. Since f is continuous and $\{ e \}$ is closed, Δ is closed as well. Since the diagonal is closed, (X, τ) is Hausdorff. $\square$

The idea around Kolmogorov spaces allow us to define the notion of points being *topologically distinguishable.* Two points are topologically indistinguishable if they belong to all of the same open sets. That is, the topology can't tell them apart. This yields an equivalence relation on the points in a topological space, and the resulting quotient is the *Kolmogorov quotient.* It always yields a Kolmogorov topological space. What's more, a topological space is a Kolmogorov space if and only if it is homeomorphic to it's Kolmogorov quotient.

In the context of groups there's a bit more to it. If $(X, \tau, *)$ is a topological group, and if $e \in X$ is the unique identity, then $\mathrm{Cl}_\tau(\{ e \})$ is a closed normal subgroup. That is, it is a closed subset as far as the topology is concerned, but it is also a normal subgroup. The points in the closure of e are precisely the ones that are topologically indistinguishable from the identity. The equivalence relation induced by being topologically indistinguishable partitions the space into the cosets of $\mathrm{Cl}_\tau(\{ e \})$ (viewed as a normal subgroup). The quotient space is hence also a topological group, and since it is Kolmogorov, it is automatically Frechét and Hausdorff. So every topological group has a canonical Hausdorff topological group associated to it. Because of this many authors require topological groups to have, as part of the definition, the Kolmogorov or Frechét or Hausdorff properties.