Math 38. Graph Theory.

Solutions to Homework 1.

Sec 1.1, # 10. It is **true**. Let G be a simple disconnected graph. Take $u, v \in V(G)$ so that there is no u, v-path in G. In particular, we have $uv \notin E(G)$, and so $uv \in E(\bar{G})$.

To show that $\bar{G}$ is connected, it is enough to show that for every vertex w, there is a path from u to w in $\bar{G}$. To see this, take an arbitrary vertex w. If $uw \in E(\bar{G})$ we are done. Otherwise, the fact that $uw \in E(G)$ and that there is no u, v-path in G forces $vw \notin E(G)$, that is, $vw \in E(\bar{G})$. But now the edges uv and vw give a path from u to w in $\bar{G}$.

Sec 1.1, # 13. **Yes**. This is solved in example 1.3.8.

Sec 1.1, # 16. **No**. Their complements are $C_4 + C_4$ (a disjoint union of two 4-cyles) and C_8, respectively, which are not isomorphic.

Sec 1.1, # 29. Consider mutual acquaintances as edges of a 6-vertex graph G (and so mutual strangers are edges of $\bar{G}$). The problem asks to show that G has either a clique of size 3 or an independent set of size 3.

Take any $v \in G$. Among the remaining 5 vertices, either we have at least 3 adjacent to v or at least 3 non-adjacent to v, by the pigeonhole principle. We can assume without loss of generality that at least 3 are adjacent to v (if we are in the other case, we argue similarly using $\bar{G}$ instead of G), that is, $vu_1, vu_2, vu_3 \in E(G)$. If one of u_1u_2, u_2u_3, u_1u_3 is in $E(G)$, then G has a triangle, consisting of the two endpoints of this edge together with v. If not, then G has an independent set $\{u_1, u_2, u_3\}$.

Sec 1.1, # 30. Let a_i denote the i-th column of the adjacency matrix A. Then the entry in row i and column i of A^2 equals $a_i^T a_i$, which is the number of ones in a_i, which is the degree of the i-th vertex v_i.

Similarly, letting b_i^T denote the i-th row of the incidence matrix M, the entry in row i and column i of MM^T equals $b_i^T b_i$, which is the number of ones in b_i, which is again degree of the v_i.

By a similar argument, the entry in position i, j of A^2 is the number of common neighbors of v_i and v_j, which equals the number of walks of length 2 from v_i to v_j.

The entry in position i, j of MM^T is the number of edges between v_i and v_j, which must be 0 or 1 for a simple graph.