Math 38. Graph Theory.

Solutions to Homework 2.

Sec 1.2, # 8. $K_{m,n}$ is Eulerian iff both m and n are even. This is because $K_{m,n}$ has m vertices of degree n and n vertices of degree m, and we have proved in class that a connected graph is Eulerian iff all its vertices have even degree.

Sec 1.2, # 10. a) **True**. The number of edges of a bipartite graph equals the sum of the degrees of the vertices in one of the two sets of the bipartition. Since the graph is Eulerian, all these degrees are even, and thus so is their sum.

b) **False**. A triangle together with an isolated vertex gives an Eulerian graph with 4 vertices and 3 edges. It is also possible to construct a connected counterexample by taking a triangle and a square with a shared vertex, giving an Eulerian graph with 6 vertices and 7 edges.

Sec 1.2, # 17. We show that there is a path between any vertex and the vertex corresponding to the permutation $123\dots n$. The proof is by induction on n.

The base case $n = 1$ is trivial, since the graph only has one vertex.

Now, assuming the result holds for $n-1$, we will prove it for n. Given a permutation $a_1 \dots a_n$, if n is not in the last position, then we can perform adjacent transpositions to move n to the last position. This gives a path in G_n from $a_1 \dots a_n$ to a permutation of the form $b_1 \dots b_{n-1}n$. By the induction hypothesis, there is a path in G_{n-1} between $b_1 \dots b_{n-1}$ and $12\dots(n-1)$, and the same operations give a path in G_n from $b_1 \dots b_{n-1}n$ to $123\dots n$. By concatenating the two paths, it follows that there is a walk (and therefore a path, as we showed in class) from $a_1 \dots a_n$ to $123\dots n$.

Sec 1.2, # 29. First we prove the following property about G:

Lemma. If $uv, vw, wx \in E(G)$, then $ux \in E(G)$.

Proof of Lemma. Consider the induced subgraph $G[\{u, v, w, x\}]$. If this subgraph contained no more edges (other than uv, vw, wx), then it would be a copy of P_4, contradicting the hypothesis. So, it must contain other edges. If it contained uw or xv, then it would have an induced copy of C_3, again a contradiction. Thus, it must contain the edge ux.

Now we prove that G is bipartite by showing that it contains no odd cycle. Suppose for contradiction that G has an odd cycle. Take such a cycle with the smallest number of edges, say ℓ. Note that $\ell \neq 1$ because G is simple, and that $\ell \neq 3$ because G contains no C_3 as an induced subgraph. Thus, $\ell \geq 5$. But if u, v, w, x are four consecutive vertices in this cycle, then $ux \in E(G)$ by the lemma, and so we could replace the piece $uvwx$ of the cycle with ux to get a cycle of length $\ell - 2$, which is a contracition with the choice of ℓ to be the smallest. It follows that G is bipartite. Let $X \cup Y$ be a partition of its vertices into independent sets.

To show that G is complete bipartite, we need to show that given any $u \in X$ and $v \in Y$, we have $uv \in E(G)$. Take a shortest path from u to v, which exists because G is connected. Let m be the length of this path, and note that m is odd because the path alternates between X and Y. If $m \geq 3$, let the first 4 vertices of the path be u, x_1, x_2, x_3. By the lemma, $ux_3 \in E(G)$. But then, deleting x_1 and x_2 from the path we would get a shorter uv-path, which contradicts minimality of m. Thus, we must have $m = 1$, and so $uv \in E(G)$.

Sec 1.2, # 38. By induction on the number of vertices n.

If $n = 1$, then the graph consists of a loop, which is a cycle of length 1.

Let now $n > 1$. Assuming the result holds for $n-1$, let us prove it for n. If all the vertices have degree ≥ 2, then G contains a cycle by Lemma 1.2.25. If some vertex has degree 1 or 0, then by deleting one such vertex we get a graph with $n-1$ vertices and (at least) $n-1$ edges. By induction hypothesis it contains a cycle, which is also a cycle in the original graph.

Sec 1.3, # 1. **True**. Suppose not. Then u and v are in different components. Let H be the component containing u. Then H is a graph with only one vertex of odd degree, in contradiction with Corollary 1.3.5.

Sec 1.3, # 12. Suppose uv is cut-edge. Removing it leaves two components, one containing u and one containing v. But then each of these components would be a graph with only one vertex of odd degree, in contradiction with Corollary 1.3.5.

For the second part, we construct a $2k+1$-regular simple graph having a cut-edge. Start with the hypercube Q_{2k+1}, and take k edges $x_iy_i \in E(Q_{2k+1})$ for $1 \leq i \leq k$ so that all $2k$ endpoints are different. Add a new vertex u, and replace each edge x_iy_i above with two edges x_iu and y_iu. At this point, all vertices have degree $2k+1$ except for u, which has degree $2k$. Call this graph G. Now take two disjoint copies of G, and add an edge connecting the vertices of degree $2k$ in each copy. This is then a cut-edge, and the resulting graph is $2k+1$-regular.

Sec 1.3, # 32. Consider the map φ from simple even graphs with vertex set $[n]$ to simple graphs with vertex set $[n-1]$ obtained by deleting the vertex n. Then φ is a bijection, since every simple graph H with vertex set $[n-1]$ has exactly one preimage, namely, the graph obtained from H by adding a new vertex n, and adding one edge between vertex n and each vertex of H of odd degree. The resulting graph G is even, and it is the only one with the property that $\varphi(G) = H$.

The number of simple graphs with vertex set $[n-1]$ is $2^{\binom{n-1}{2}}$, because for each pair of vertices, we can choose to include an edge between them or not, giving $\binom{n-1}{2}$ independent choices. Since φ is a bijection between this set and the set of even graphs with vertex set $[n]$, the two sets have the same cardinality.