Math 38. Graph Theory.

Solutions to Homework 3.

Sec 1.3, # 8. For parts a), b), c) below, ask me if you want to see the construction of the simple graph with the degree sequence.

a) 5543221 is graphic by Theorem 1.3.31:

$$5543221 \leftrightarrow 4321121 \sim 4322111 \leftrightarrow 211011 \sim 211110 \leftrightarrow 11000.$$

b) 55442211 is graphic by Theorem 1.3.31:

$$55442211 \leftrightarrow 4331111 \leftrightarrow 220011 \sim 221100 \leftrightarrow 10100$$

c) 55532211 is graphic by Theorem 1.3.31:

$$55532211 \leftrightarrow 4421111 \leftrightarrow 310011 \sim 311100 \leftrightarrow 00000$$

d) 55542111 is not graphic by Theorem 1.3.31:

$$55542111 \leftrightarrow 4431011 \sim 4431110 \leftrightarrow 320010 \sim 321000,$$

but the last sequence is not graphic, since the vertex of degree 3 can't find enough neighbors.

Sec 1.4, # 9. **False**. Here is a counterexample. Let $V(D) = \{1, 2, \dots, n\}$, and let

$$E(D) = \{(i, j) : 1 \le i < j \le n\}.$$

Then all the outdegrees are different, and so are the indegrees.

Sec 1.4, # 10. First we prove the forward direction. Suppose that there is some partition into S and T such that there is no edge from S to T. Then there is no path from vertices in S to vertices in T, so the graph can't be strongly connected.

For the converse, we assume that the condition on the partitions holds and we want to show that for every ordered pair u, v of vertices, there is a path from u to v. Let S be the set of vertices x such that there is a path from u to x. Let T be the set of vertices not in S. We will show that T is empty, and so $v \in S$ and we are done. Indeed, if T wasn't empty, then we know that there is an edge from S to T. But then there is a path from u to the head of that edge, contradicting that the head was not in S.

Sec 1.4, # 15. Consider the vertices as points in the plane given by their coordinates. Each path from $(0, 0)$ to (m, n) in the digraph uses m horizontal edges and n vertical edges. The path is uniquely determined by choosing, among its $m + n$ edges, which ones are horizontal and which ones are vertical. Thus there are $\binom{m+n}{n}$ choices.