Math 38. Graph Theory.

Solutions to Homework 6.

3.1.5. Let S be a maximum independent set. We claim that every vertex of G is either in S, or it is adjacent to a vertex in S. Indeed, if this was not the case for some vertex, then we could add that vertex to S to get a larger independent set, contradicting the fact that S was maximum. It follows that

$$V(G) \subseteq \bigcup_{v\in S}(N(v)\cup\{v\}).$$

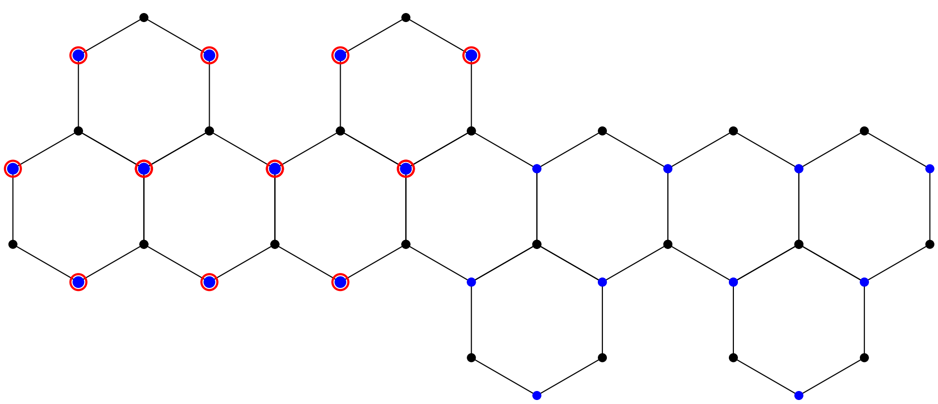
Using that $|N(v)\cup\{v\}| \le \Delta(G)+1$ for any v, we get

$$n(G) = |V(G)| \le \sum_{v\in S}|N(v)\cup\{v\}| \le |S|(\Delta(G)+1) = \alpha(G)(\Delta(G)+1).$$

3.1.9. Let M be a maximal matching of G, and let M' be a maximum matching, so that $|M'| = \alpha'(G)$. By Lemma 3.1.9, the components of $M \triangle M'$ are alternating paths and even cycles. If such a component consisted of a single edge in $M' \setminus M$, then that edge could be added to M to create another matching, contradicting the maximality of M.

Thus, all the components of $M\triangle M'$ are alternating paths of length at least 2, or even cycles. In any such component, the number of edges in $M' \setminus M$ is at most twice the number of edges in $M \setminus M'$. It follows that $|M'| \le 2|M|$, and so $|M| \ge |M'|/2 = \alpha'(G)/2$.

3.1.28. This graph is bipartite, since we can explicitly partition the vertices into two independent sets X and Y, which have the same size. In the picture below, the vertices in X are blue, and the ones in Y are black.



By Hall's theorem (Thm 3.1.11), to show that this graph has no perfect matching (equivalently, no matching that saturates X), it is enough to find a subset $S \subseteq X$ with $|N(S)| < |S|$. The set S of red circled vertices in the graph satisfies $|S| = 11$ and $|N(S)| = 10$.

4.1.5. We will show that if $x \in V(G')$, then $G' - x$ is connected. Let $u, v \in V(G'-x)$. We want to find a u,v-path in $G'-x$. Let P be a u,v-path in G. If x is not a vertex in P, then we are done. Otherwise, let y be the vertex preceding x in P (in the direction from u to v) and let z be the vertex following x. Since $d_G(y,z) \le 2$, we have that $yz \in E(G')$. Replacing yx and xz in P with yz we obtain a u,v-path in $G'-x$.