Math 38. Graph Theory.

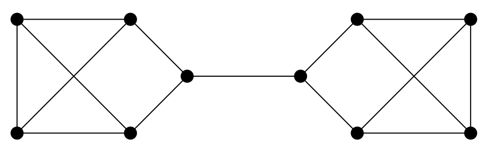
Solutions to Homework 7.

4.1.10. By Theorem 4.1.11, we must have $\kappa(G) = \kappa'(G) = 1$. Thus, G has a cut-edge, say uv. Let H be the component of $G - e$ that contains u. Since G is 3-regular, u has two more neighbors (other than v), say x and y. Since $d(x) = 3$, H must have at least 4 vertices, namely x and its three neighbors.

Let $m \geq 4$ be the number of vertices of H, and let $e(H)$ be its number of edges. By the degree-sum formula, $2e(H) = 3m - 1$, since all vertices of H other than u have degree 3, and u has degree 2 in H. This formula implies that m is odd, and so $m \geq 5$. Additionally, if $m = 5$, the two vertices in H other than u, x, y must be connected to each other and to x, y in order to have degree 3.

The same argument applies to the component of $G - e$ that contains v. Thus, G must have at least 10 vertices.

Here is a 3-regular graph with 10 vertices having connectivity 1:



By the above argument, it is the smallest one.

4.1.11. The proof is almost the same as that of Theorem 4.1.11. Let S be a minimum vertex cut. The fact that every $v \in S$ must have at least a neighbor in each of H_1 and H_2 still holds. So does the fact that v cannot have two neighbors in H_1 and two neighbors in H_2, since $\Delta(G) \leq 3$. Thus, for each $v \in S$, if v has only one edge to H_1, we delete it; else, v has only one edge to H_2, so we delete that one. This gives an edge cut of size $|S| = \kappa(G)$.

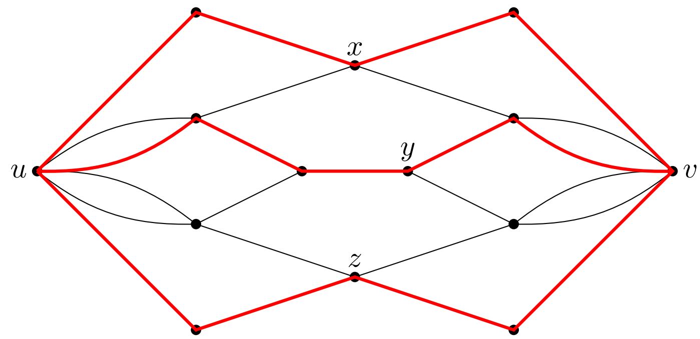
4.1.15. The Petersen graph G is 3-regular. By Theorem 4.1.11, $\kappa(G) = \kappa(G')$. Thus, to show that G is 3-connected, it suffices to show that $\kappa'(G) \geq 3$, that is, G has no edge cut of size 2 or less. Clearly, G has no cut-edge. Suppose for contradiction that it has an edge cut $[S, \overline{S}]$ of size 2. Then, by Prop 4.1.12,

$$2 = \big|[S, \overline{S}]\big| = \sum_{v \in S} d(v) - 2e(G[S]) = 3|S| - 2e(G[S]).$$

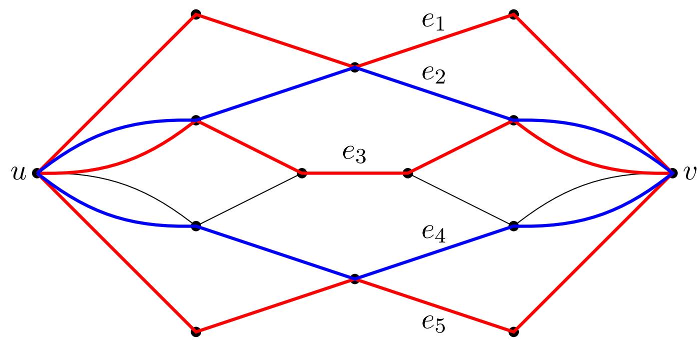
In particular, this implies that $|S|$ is even. Also, by switching S with $\overline{S}$ if necessary, we can assume without loss of generality that $|S| \leq |\overline{S}| = 10 - |S|$, so $|S| \leq 5$.

If $|S| = 2$, then the above formula would imply $e(G[S]) = 2$, so G would have a multiple edge, which we know is not true. So, the only case left is $|S| = 4$, which implies $e(G[S]) = 5$. This means that $G[S]$ is the complete graph K_4 (which has 6 edges) minus an edge. But then $G[S]$ would have a triangle, which contradicts the fact that the Petersen graph has no triangles (as a triangle would require three pairwise-disjoint 2-element subsets of [5]).

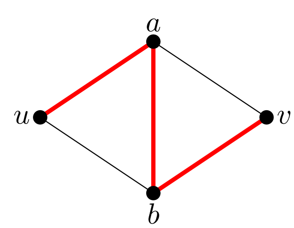
4.2.1. We know that $\kappa(u,v) \leq 3$, because deleting the vertices x, y, z (see picture) separates u from v. On the other hand, $\kappa(u,v) \geq \lambda(u,v) \geq 3$, because there are 3 pairwise internally disjoint u,v-paths, drawn in red. We conclude that $\kappa(u,v) = 3$.



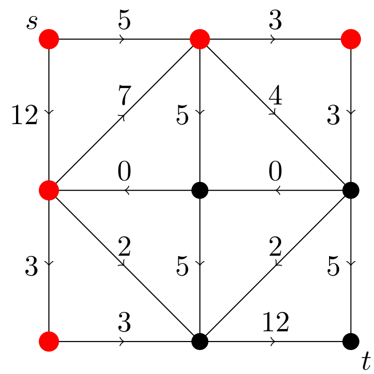
We know that $\kappa'(u,v) \leq 5$, because deleting the edges e_1, e_2, e_3, e_4, e_5 (see picture) disconnects u from v. On the other hand, $\kappa'(u,v) \geq \lambda'(u,v) \geq 5$, because there are 5 pairwise edge-disjoint u,v-paths, drawn in red and blue. We conclude that $\kappa'(u,v) = 5$.



4.2.4. This is false. Here is a counterexample. The graph below is 2-connected, and $P = uabv$ is a u,v-path, but there is no u,v-path Q internally disjoint from P.



4.3.2. Using the Ford-Fulkerson Algorithm, we get the flow f drawn below:



Its value is $\operatorname{val}(f) = 17$. Additionally, it is a maximum flow because, letting S be the red vertices, tha capacity of the resulting source/sink cut is $\operatorname{cap}(S, \overline{S}) = 3 + 4 + 5 + 2 + 3 = 17$, which agrees with the value of the flow f.

4.3.10. For every graph G, the inequality $\alpha'(G) \leq \beta(G)$ holds because a vertex can't cover two edges of a matching. Now let G be an X, Y-bipartite graph. We will prove that $\alpha'(G) \geq \beta(G)$.

Let D be the network whose nodes are $V(G) \cup \{s, t\}$, and whose edges are

$$\begin{cases} s \to x & \text{for each } x \in X, \\ x \to y & \text{for each } x \in X, y \in Y \text{ such that } xy \in E(G), \\ y \to t & \text{for each } y \in Y. \end{cases}$$

All the edges have capacity 1.

First, let us show that $\alpha'(G)$, which is the maximum size of a matching in G, equals the maximum value of a flow in D. This is because a matching M with m edges, say $x_i y_i \in M$ for $1 \leq i \leq m$ (where $x_i \in X$ and $y_i \in Y$), determines a flow f of value m by letting $f(sx_i) = 1$, $f(x_i y_i) = 1$ and $f(y_i t) = 1$ for $1 \leq i \leq m$, and $f(e) = 0$ for all other edges. Conversely, an integral maximum flow (which exists by Corollary 4.3.12) of value m determines a matching of size m by taking the edges from X to Y with flow 1.

By the Max-Flow Min-Cut Theorem, the maximum value of a flow equals the minimum capacity of a source/sink cut. Let $[S, T]$ be a minimum source/sink cut of D. We will produce a vertex cover of G with at most $\operatorname{cap}(S, T)$ vertices. This will prove that $\beta(G) \leq \operatorname{cap}(S, T) = \alpha'(G)$.

Define the sets

$$\begin{aligned} A &= X \cap T, \\ B &= Y \cap S, \\ C &= \{y \in Y : y \text{ is the head of an edge in } [S, T]\}, \end{aligned}$$

and let $U = A \cup B \cup C$. It remains to show that U is a vertex cover of G, and that $|U| \leq \operatorname{cap}(S, T) = |[S, T]|$.

All the edges of G either have an endpoint in A or B, or they go between $X \cap S$ and $Y \cap T$, in which case they have an endpoint in C. This shows that U is a vertex cover of G.

Finally, to show that $|U| \leq |[S,T]|$, we give an injection from U to $[S,T]$ as follows:
to each $v \in A$, we associate the edge $sv \in [S,T]$,
to each $v \in B$, we associate the edge $vt \in [S,T]$,
to each $v \in C$, we associate an edge in $[S,T]$ with head v.