Math 38. Graph Theory

Solutions to Homework 8.

4.3.14. We build a network D as follows.

There is a node u_i for each of academic department i (recall that there are k academic departments), a node v_j for each professor j, three nodes w_r for $r \in \{\text{assistant,associate,full}\}$, plus a source s and a sink t.

The edges of the network are:

- su_i for all i, with capacity 1;
- $u_i v_j$ for each professor j belonging to department i, with capacity 1;
- $v_j w_r$ for each professor j at rank r, with capacity 1;
- $w_r t$ for all r, with capacity $k/3$.

There is a committee satisfying the requirements if and only if the network D has a flow with value k. From an integral flow f of value k, the committee is obtained by choosing those professors j such that $f^+(v_j) = 1$.

5.1.12. FALSE. As a counterexample, consider the tree G with 6 vertices, consisting of one edge uv plus two leaves adjacent to u and two leaves adjacent to v. Then $\chi(G) = 2$ because trees are bipartite, and $\alpha(G) = 4$ because the 4 leaves of G form an independent set. However, the only partition of the vertices of G into two independent sets consists of two sets of size 3, so a 2-coloring cannot have a color class with 4 vertices.

5.1.14. TRUE. Pick an independent set of size $\alpha(G)$ and color its vertices with color 1. Then color each one of the $n - \alpha(G)$ remaining vertices with different colors $2, 3, \dots, n - \alpha(G) + 1$. This gives a proper $(n - \alpha(G) + 1)$-coloring.

5.1.33. Let $k = \chi(G)$. Consider a proper k-coloring $f : V(G) \to \{1, 2, \dots, k\}$ such that the sum of the colors of the vertices $\sum_{v \in V(G)} f(v)$ is minimum among all proper k-colorings. Now, order the vertices so that all vertices colored 1 by f are listed first, followed by all the vertices colored 2, then the vertices colored 3, etc. We will show that, with this ordering, the greedy algorithm will produce coloring f.

To see this, we claim that when the greedy algorithm gets to a vertex v with $f(v) = c$, then color c is available for that vertex (because f is proper), but no color c' with $c' < c$ is available for v, and so the algorithm chooses c for that vertex. Indeed, suppose for contradiction that color c' (with $c' < c$) was available for v. Then the coloring f' defined by $f'(v) = c'$ and $f'(u) = f(u)$ for all $u \neq v$ would still be a proper k-coloring, since all the vertices incident to v that haven't been colored at this point of the algorithm receive colors $\geq c$. But f' has a smaller color sum than f, contradicting the choice of f.

5.3.1. For the graph G on the left, we have $\chi(G; k) = k(k-1)^2(k-2)$. This is because we have $k(k-1)(k-2)$ ways to color the triangle, and then we can color the vertex of degree 1 with any color except for the one of its neighbor, so we have $k - 1$ colors for it.

For the graph G' on the right, we have $\chi(G';k) = k(k-1)(k-2)(k^2-3k+3)$. This because we first have $k(k-1)(k^2-3k+3)$ ways to color the 4-cycle (using $\chi(C_4;k)$ as given by Example 5.3.5), and then $k-2$ ways to color the remaining vertex, since its color has to be different than its two neighbors.

5.3.3. Plugging in $k = 2$ we get $2^4 - 4 \cdot 2^3 + 3 \cdot 2^2 = -4 < 0$, which is impossible for a chromatic polynomial.

5.3.5. We will use induction on n. For $n = 1$, $G_n = K_2$, and $\chi(G_1;k) = k(k-1)$, which agrees with the given formula.

For the induction step, let $n > 1$ and assume that $\chi(G_{n-1};k) = (k^2-3k+3)^{n-2}k(k-1)$ (induction hypothesis). Let e be the rightmost vertical edge of G_n. By the chromatic recurrence, $\chi(G_n;k) = \chi(G_n - e;k) - \chi(G_n \cdot e;k)$.

To compute $\chi(G_n - e;k)$, note that a k-coloring of $G_n - e$ can be found by first coloring G_{n-1} and then coloring each one of the two additional leaves with $k-1$ colors each, giving $\chi(G_n - e;k) = \chi(G_{n-1};k)(k-1)^2$.

On the other hand, to compute $\chi(G_n \cdot e;k)$, note that a k-coloring of $G_n \cdot e$ can be found by first coloring G_{n-1} and then coloring the additional vertex with $k-2$ colors (since it has two neighbors that have received 2 different colors already), giving $\chi(G_n \cdot e;k) = \chi(G_{n-1};k)(k-2)$.

It follows that

$$\begin{aligned}\chi(G_n;k) &= \chi(G_n - e;k) - \chi(G_n \cdot e;k) = \chi(G_{n-1};k)[(k-1)^2 - (k-2)] \\ &= \chi(G_{n-1};k)(k^2-3k+3) = (k^2-3k+3)^{n-1}k(k-1),\end{aligned}$$

using the induction hypothesis in the last equality.

5.1.25. (Bonus) Come to my office hours to ask me if you want to see a solution to this problem.