Characterizations of trees

Theorem

Let G be a graph with n vertices. Then the following are equivalent:

A) *G is a tree (i.e., it is connected and has no cycles).*

B) *G is connected and has $n-1$ edges.*

C) *G has $n-1$ edges and no cycles.*

D) *For any $u, v \in V(G)$, there is exactly one u, v-path.*

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Some consequences:

- Every edge of a tree is a cut-edge.
- Adding an edge to a tree forms exactly one cycle.

Spanning trees

Definition

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Distance in trees and graphs

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Definition

The **center** of G is the subgraph induced by the vertices of minimum eccentricity.