Comparing matchings

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Theorem (Berge '57)

A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.

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Corollary

For $k > 0$, every k-regular bipartite graph has a perfect matching.