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We will show that, in some cases, there is an easier way.

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$\beta(G) =$ minimum size of a vertex cover in G
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Obtaining a matching and a vertex cover of the same size proves that both are optimal!

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This is an example of a min-max relation.

In general, we have a maximization problem and a minimization problem, and any solution to the first has a smaller size than any solution to the second.

If we find solutions with the same value, then they are both optimal.

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$$\beta'(G) \geq \frac{n(G)}{2} \geq \alpha'(G)$$