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For general *n* and $k \ge 2$, there is an *n*-vertex graph $H_{n,k}$, called the Harary graph, which has

$$\kappa(H_{k,n}) = k$$
 and $e(H_{k,n}) = \left\lceil \frac{kn}{2} \right\rceil$.

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For every graph G,

 $\kappa(G) \leq \kappa'(G) \leq \delta(G).$

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If G is 3-regular, then

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