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For general n and $k \geq 2$, there is an n-vertex graph $H_{n,k}$, called the **Harary graph**, which has

$$\kappa(H_{k,n}) = k \quad \text{and} \quad e(H_{k,n}) = \left\lceil \frac{kn}{2} \right\rceil.$$

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Whitney's Theorem

Theorem (Whitney '32)

For every graph G,

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If G is 3-regular, then

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