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Theorem

If G *is 3-regular, then*

$$\kappa(G) = \kappa'(G).$$

More about edge cuts

Proposition

If $S \subseteq V(G)$, *then*

$$\big|[S, \overline{S}]\big| = \sum_{v \in S} d(v) - 2e(G[S]).$$

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An edge cut may contain another edge cut.

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A **block** of G is a maximal connected subgraph that has no cut-vertex.

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Proposition

Two blocks in a graph share at most one vertex.

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A graph G is 1-connected (a.k.a. connected) if and only if, for every $u, v \in V(G)$, there is a u, v-path.

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Theorem

A graph is 2-connected if and only if it has an "ear decomposition".

x, y-cuts

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In fact,

$$\kappa(G) = \min_{x,y \in V(G)} \kappa(x, y).$$