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If G is 3-regular, then

$$\kappa(G) = \kappa'(G).$$

More about edge cuts

Proposition

If $S \subseteq V(G)$, then

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An edge cut may contain another edge cut.

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Definition

A **block** of G is a maximal connected subgraph that has no cut-vertex.

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Proposition

Two blocks in a graph share at most one vertex.

4.2 k-connected graphs

A graph G is 1-connected (a.k.a. connected) if and only if, for every $u, v \in V(G)$, there is a u, v-path.

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A graph is 2-connected if and only if it has an "ear decomposition".

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In fact,

$$\kappa(G) = \min_{x,y \in V(G)} \kappa(x, y).$$

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Corollary

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Given $x, y \in V(G)$, let

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The line graph

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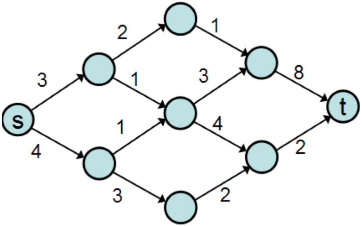
Definition

The **line graph** of G, denoted by $L(G)$, is the graph whose vertices are the edges of G, and with $ef \in E(L(G))$ if e and f are edges of G that share an endpoint.

4.3 Network flow problems



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- Edges represent pipes where water flows in one direction, and the labels indicate their capacity (amount of water per second). What is the maximum flow from s to t?

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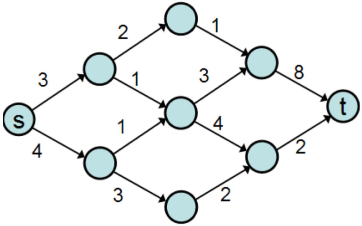

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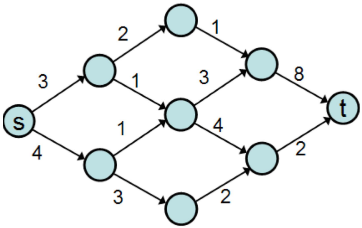

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- The diagram represents a computer (or electrical) network, and the labels indicate the data (or electricity) transmission capacities. How much data (or current) can be transmitted from s to t?

Networks

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Each edge has a nonnegative label $c(e)$, called its **capacity**.

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A **feasible flow** is a function $f : E(D) \to \mathbb{R}_{\geq 0}$ that assigns to each edge a non-negative real number, such that

- $0 \leq f(e) \leq c(e)$ for all $e \in E(D)$; (capacity constraints)

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- $0 \leq f(e) \leq c(e)$ for all $e \in E(D)$; (capacity constraints)
- $f^+(v) = f^-(v)$ for all $v \in V(G) \setminus \{s, t\}$,
  where $f^+(v)$ is the flow on edges leaving v,
  and $f^-(v)$ is the flow on edges entering v.
  (conservation constraints)