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In fact,

$$\kappa(G) = \min_{x,y \in V(G)} \kappa(x, y).$$

Menger's theorem

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If $x, y \in V(G)$ and $xy \notin E(G)$, then

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If $x, y \in V(G)$ and $xy \notin E(G)$, then

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Corollary

A graph G is k-connected if and only if, for every $u, v \in V(G)$, there are at least k internally disjoint u, v-paths.

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Definition

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The line graph

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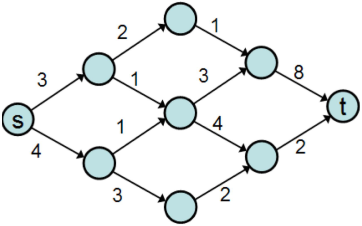
Definition

The **line graph** of G, denoted by $L(G)$, is the graph whose vertices are the edges of G, and with $ef \in E(L(G))$ if e and f are edges of G that share an endpoint.

4.3 Network flow problems



Examples:

- Edges represent pipes where water flows in one direction, and the labels indicate their capacity (amount of water per second). What is the maximum flow from s to t?

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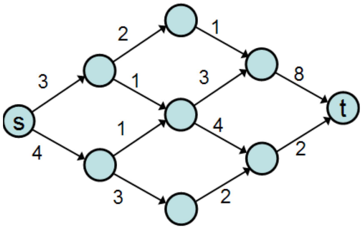

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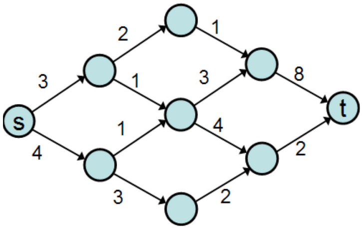

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- The diagram represents a computer (or electrical) network, and the labels indicate the data (or electricity) transmission capacities. How much data (or current) can be transmitted from s to t?

Networks

Definition

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- $0 \leq f(e) \leq c(e)$ for all $e \in E(D)$; (capacity constraints)
- $f^+(v) = f^-(v)$ for all $v \in V(G) \setminus \{s, t\}$,
  where $f^+(v)$ is the flow on edges leaving v,
  and $f^-(v)$ is the flow on edges entering v.
  (conservation constraints)

Maximum flow

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The **value** of a flow f, denoted by $\mathrm{val}(f)$, is the net flow into the sink, that is,

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A **maximum flow** is a feasible flow of maximum value.

f-augmenting paths

Definition

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How can we know when our flow is maximum?